A Preferred-Habitat Model of Term Premia, Exchange Rates, and Monetary Policy Spillovers

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January 21, 2025*

Abstract

We develop a two-country model in which currency and bond markets are populated by different investor clienteles, and segmentation is partly overcome by arbitrageurs with limited capital. Risk premia in our model are time-varying, connected across markets, and consistent with the empirical violations of Uncovered Interest Parity and Expectations Hypothesis. Through risk premia, large-scale bond purchases lower domestic and foreign bond yields and depreciate the currency, and short-rate cuts lower foreign yields, with smaller effects than bond purchases. Currency returns are disconnected from long-maturity bond returns, and yet the currency market is instrumental in transmitting bond demand shocks across countries.

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1 Introduction

How is monetary policy transmitted domestically and internationally? The standard international macroeconomics model with perfect capital mobility and floating exchange rates (e.g., Gali (2015)) delivers sharp answers. These follow from the Expectations Hypothesis (EH) and the Uncovered Interest Parity (UIP), which hold in the standard model up to constant risk premia. Because of EH, the yield curve in each country depends only on expectations of the domestic short rate, which is controlled by the domestic central bank. Hence, Quantitative Easing or Tightening (QE/QT) by the central bank, keeping short rates unchanged, has no effect on the yield curve. Moreover, each country's yield curve is fully insulated from other countries' monetary policy. Insulation arises because according to UIP, short-rate differentials across countries are absorbed into the exchange rate, whose expected movements compensate investors for these differentials. The insulation result is a slightly broader statement of the well-known Friedman-Obstfeld-Taylor Trilemma: with perfect capital mobility, a floating exchange rate provides monetary autonomy not just in setting short rates but also in shaping the entire yield curve.[1]

Four broad empirical observations cast doubt on the validity of the standard model. First, a large literature starting with Bilson (1981) and Fama (1984) documents strong violations of UIP: currency carry trade (CCT) strategies that borrow in currencies with low short rates and invest in currencies with high short rates earn abnormally high expected returns. Second, a similarly large literature starting with Fama and Bliss (1987) and Campbell and Shiller (1991) documents strong violations of EH: bond carry trade (BCT) strategies that borrow in maturities with low interest rates and invest in maturities with high interest rates earn abnormally high expected returns. Third, risk premia in currencies and bonds are connected. For example, Chen and Tsang (2013), Lloyd and Marin (2020) and Chernov and Creal (2021) find that yield curve slope differentials predict the CCT's profitability, and Lustig, Stathopoulos, and Verdelhan (2019) find that the CCT's profitability declines when that trade is carried out with long-maturity rather than short-maturity bonds.[2] Fourth, a growing body of evidence surveyed in Bhattarai and Neely (2022) suggests that central banks' QE purchases had a significant impact not only on domestic yields but also on exchange rates and foreign yields.[3]

In this paper we develop a two-country model in which currency and bond markets are populated by different investor clienteles, and segmentation is partly overcome by global arbitrageurs with limited capital. We show that the combination of limited arbitrage

[1] Obstfeld, Shambaugh, and Taylor (2010) provide a modern articulation of the Trilemma.

[2] See Clarida and Taylor (1997) and Berge, Jorda, and Taylor (2010) for related evidence that the home and foreign yield-curve factors predict the CCT's profitability, and Chernov and Dahlquist (2023) for a literature survey on currency risk premia emphasizing their connection with bond risk premia.

[3] See also Bauer and Neely (2014), Neely (2015), Curcuru, De Pooter, and Eckerd (2018), Fratzscher, Lo Duca, and Straub (2018), Dedola, Georgiadis, Grab, and Mehl (2021), Curcuru, Kamin, Li, and Rodriguez (2023) and Georgiadis and Jarocinski (2023).

and price-elastic clienteles gives rise to the empirically documented violations of UIP and EH, and to the ways in which the violations are connected. The resulting risk premia in currency and bond markets, which are time-varying and connected, play a key role in the transmission of monetary policy. It is through changes in risk premia in both markets that QE purchases lower domestic and foreign bond yields and depreciate the currency, and that short-rate cuts lower foreign bond yields. These effects are absent from the standard model, in which risk premia are constant.

The connections between currency and bond risk premia that arise in our model are consistent with the low empirical correlation between currency and long-maturity bond returns. In our estimated model, movements in the exchange rate are almost uncorrelated with movements in long-maturity bond yields, and yet the currency market is instrumental in transmitting bond demand shocks across countries. In particular, the effects of QE/QT on foreign bond yields are sizeable, and stronger than those of conventional policy. At the same time, because of the disconnect between the exchange rate and bond yields, the transmission of QE/QT to the exchange rate is weaker, and foreign exchange interventions by central banks have strong effects on the exchange rate but weak effects on bond yields.

Our model, presented in Section 2, is set in continuous time and infinite horizon. In each of the two countries, home and foreign, a central bank sets the short rate exogenously. There are three types of agents: currency traders, bond investors, and global arbitrageurs. Currency traders express a demand for foreign assets. Bond investors form clienteles, each of which expresses a demand for a bond of a specific country and maturity. Currency traders' demand can depend on the real exchange rate, and bond investors' demand can depend on bond prices. Both types of demands are exogenous and subject to shocks: a currency demand factor shifts the demand of currency traders and a bond demand factor in each country shifts the demand of that country's bond investors. Examples of currency traders are households switching expenditure between home and foreign goods. Examples of bond clienteles are pension funds and insurance companies.

The existence of currency traders and bond investors introduces segmentation across asset classes and maturities. Segmentation is partly overcome by global arbitrageurs, who trade the currency and bonds of both countries. Arbitrageurs maximize mean-variance utility over instantaneous changes in wealth. Their risk aversion parameter can capture in reduced form capital or Value-at-Risk constraints. Examples of global arbitrageurs are macro hedge funds and global banks.

Section 3 characterizes the equilibrium as a solution to a scalar non-linear system. The exchange rate and bond prices are log affine functions of five stochastic factors: the two short rates, the currency demand factor, and the two bond demand factors. When arbitrageur risk aversion is zero, UIP and EH hold, and monetary policy transmission is as in the standard model.

Section 4 specializes the model to the case where short rates are the only stochastic factors and are independent across countries. In that case, we can show key mechanisms and results analytically. Consider the transmission of conventional monetary policy. Following a cut to the home short rate, arbitrageurs find it attractive to enter into the CCT, by borrowing in the home currency and investing in the foreign currency. If the demand by currency traders is price-elastic, then arbitrageurs' holdings of foreign currency rise in equilibrium. The expected return of the CCT rises as well, as arbitrageurs must be compensated for the risk of entering into that trade. The rise in the CCT's expected return attenuates the transmission of monetary policy to the exchange rate, which appreciates less than implied by UIP. This attenuation effect parallels Gabaix and Maggiori (2015), who model exchange-rate dynamics without a yield curve.

A similar attenuation effect arises in the home bond market. The short-rate cut prompts arbitrageurs to also enter into the home BCT, by borrowing in the home short rate and investing in long-maturity home bonds. If the demand by home bond investors is price-elastic, then arbitrageurs' holdings of home bonds rise in equilibrium and so does the expected return of the home BCT. The rise in the BCT's expected return attenuates the transmission of monetary policy to domestic bond yields, which drop less than implied by EH. This attenuation effect parallels Vayanos and Vila (2021), who model yield-curve dynamics in a closed economy.

In addition to the above attenuation effects, we show a propagation effect of conventional policy to foreign bond yields. Propagation occurs because of hedging by arbitrageurs. By entering into the CCT in response to the home short-rate cut, arbitrageurs become more exposed to the risk that the foreign short rate drops and the foreign currency depreciates. Foreign bonds provide a natural hedge for that risk because their price rises when the foreign short rate drops. Hence, arbitrageurs increase their demand for foreign bonds, causing foreign bond yields to drop.

Consider next the transmission of unconventional monetary policy. Following QE purchases of home bonds, their prices rise. Arbitrageurs accommodate the increased demand for home bonds by holding fewer such bonds. This renders them less exposed to a rise in the home short rate and more willing to hold foreign currency, which depreciates when the home short rate rises. Arbitrageurs also become more willing to hold foreign bonds, which hedge the foreign currency position against a drop in the foreign short rate. Hence, QE purchases depreciate the home currency and lower foreign bond yields. A similar argument implies that sterilized purchases of foreign currency by the home or foreign central bank lower home bond yields and raise foreign ones.

Section 5 solves the full model and quantifies its effects. The model parameters are estimated from data on the US and the Eurozone, using maximum likelihood. Demand shocks generate a disconnect between the exchange rate and long-maturity bond yields, but not between home and foreign bond yields. Variance decompositions within our estimated model reveal that a country's long-maturity bond yields move primarily because of the bond demand factors in both countries. Short rates account for a small fraction of movements in long-maturity yields, and the currency demand factor for an even smaller one. Conversely, the currency demand factor is the exchange rate's primary driver. Short rates account for a small fraction of exchange-rate movements, and the bond demand factors for an even smaller one.

The result that long-maturity bond yields are connected across countries despite their disconnect with the exchange rate can seem puzzling. This is because Section 4 shows that shocks to bond demand in one country are transmitted to bond yields in the other through the hedging of arbitrageurs' bond positions in the currency market. In Section 4, the exchange rate and bond yields move only because of the short rates, and bond yields comove positively across countries. Introducing the bond demand factors strengthens the comovement between home and foreign yields—even though the factors themselves are independent across countries. This is because bond demand shocks in one country move yields in both countries in the same direction. On the other hand, the comovement between bond yields and the exchange rate is unaffected because bond demand shocks in the two countries move the exchange rate in opposite directions. As a result, the exchange rate is connected weakly to long-maturity bond yields and strongly to the short rates. Introducing the currency demand factor weakens the latter connection as well. Because currency demand shocks generate opposite movements in home and foreign bond yields, as shown in Section 4, they weaken the strong positive comovement between long-maturity bond yields across countries, but they do not undo it. As a result, our model features simultaneously positive comovement in bond yields across countries, transmission of bond demand shocks through the currency market, and exchange-rate disconnect.

Our estimated model matches closely the violations of UIP and EH found in the literature and the ways in which the violations are connected. It also generates sharp implications on the effects of monetary policy and foreign exchange interventions. These follow from the exchange-rate disconnect and from the strong positive comovement between bond yields across countries.

Our paper is part of a recent literature that emphasizes the role of segmented markets, financial intermediaries and limits of arbitrage for macroeconomics. In Gabaix and Maggiori (2015), households in each of two countries can only invest in a domestic bond, while intermediaries can invest in the bonds of both countries. Because intermediary positions are constrained, UIP fails to hold, and the exchange rate is influenced by financial flows as in an earlier literature on portfolio balance (e.g., Kouri (1976) and Driskill and McCafferty (1980)). In Itskhoki and Mukhin (2021), the exchange rate is determined by households who can only invest in a domestic bond, risk-averse intermediaries who can overcome this segmentation, and noise traders. Shocks to noise-trader demand generate UIP deviations, and account for more than 90% of exchange rate fluctuations but for only a small fraction of output fluctuations. In Kekre and Lenel (2024), the exchange rate correlates more highly with output because of demand shocks that shift households' discount rates. Other frictional models of exchange rates with financial flows and noise traders include Jeanne and Rose (2002), Evans and Lyons (2002), Hau and Rey (2006), Bacchetta and van Wincoop (2010, 2021) and Bruno and Shin (2015). These papers focus on the determination of the exchange rate and do not model yield curve frictions.

Vayanos and Vila (2021, VV) develop a closed-economy model of the yield curve with preferred-habitat investors who can invest in specific maturities and risk-averse arbitrageurs. Our model extends VV to two countries, adds the currency market, and assumes global arbitrageurs. Ray (2019) and Ray, Droste, and Gorodnichenko (2024) embed VV into a New Keynesian model and study the transmission of conventional and unconventional monetary policy to the domestic economy. Closest to our work is Greenwood, Hanson, Stein, and Sunderam (2023), who develop independently a discrete-time model of currency and bond markets with preferred-habitat investors, global arbitrageurs, and only two bond maturities. They derive results analogous to those that we present in Section 4 and explore additionally Covered Interest Parity violations and arbitrageur heterogeneity, but do not estimate their model or quantify its effects.

Our paper is also related to a recent literature that examines how convenience yields affect exchange rates and interest rates. Jiang, Krishnamurthy, and Lustig (2021) and Engel and Wu (2022) construct convenience yields by comparing home government bonds to synthetic counterparts constructed by buying foreign government bonds and swapping the foreign into the home currency. They find that the home currency appreciates when the home convenience yield rises. Devereux, Engel, and Wu (2023) and Jiang, Krishnamurthy, and Lustig (2024) show that investor preferences for safe dollar assets underlie the global financial cycle (Rey, 2013) whereby US monetary policy transmits to the rest of the world. Our model can capture investor preferences for currencies and bonds of a specific country through the demand factors, and can quantify the effects of each type of demand.

Finally, our paper is related to DSGE models of monetary policy transmission. Closest to our work is the two-country model of Alpanda and Kabaca (2020). They find that QE purchases have large international spillover effects, which exceed those of conventional monetary policy. Portfolio balance effects in their model arise from bond holdings entering directly in agents' utility functions, while we partly endogenize them through mean-variance optimization by arbitrageurs.[4]

[4]See also Andres, Lopez-Salido, and Nelson (2004) and Chen, Curdia, and Ferrero (2012), for closed-economy DSGE models of QE in which agents face transaction costs and position limits when trading bonds.

2 Model

Time is continuous and goes from zero to infinity. There are two countries, home (H) and foreign (F). We define the exchange rate as the units of home currency that one unit of foreign currency can buy, and denote it by e_t at time t. An increase in e_t corresponds to a foreign currency appreciation.

In each country $j = H, F$, a continuum of zero-coupon government bonds can be traded. The bonds' maturities lie in an interval $(0, T)$. The country-j bond with maturity τ at time t pays off one unit of country j's currency at time $t + \tau$. We denote by $P_{jt}^{(\tau)}$ the time-t price of that bond, expressed in units of country j's currency, and by $y_{jt}^{(\tau)}$ the bond's yield. The yield is the spot rate for maturity τ, and is related to the price through

$$y_{jt}^{(\tau)} = -\frac{\log\left(P_{jt}^{(\tau)}\right)}{\tau}. \tag{2.1}$$

The country-j and time-t short rate i_{jt} is the limit of the yield $y_{jt}^{(\tau)}$ when τ goes to zero. We take i_{jt} as exogenous, and describe its dynamics later in this section (Equation (2.7)). An exogenous i_{jt} can be interpreted as the result of actions that the central bank in country j takes when targeting the short nominal rate by elastically supplying liquidity.

There are three types of agents: arbitrageurs, currency traders, and bond investors. Arbitrageurs are competitive and maximize a mean-variance objective over instantaneous changes in wealth. For tractability, we express the wealth of all arbitrageurs in units of the home currency, thus assuming that the home currency is the riskless asset for them. Assuming that for some arbitrageurs the foreign currency is the riskless asset renders our model nonlinear, but its linearized version remains the same.

We allow arbitrage to be global or segmented. When arbitrage is global, arbitrageurs can invest in the currency and bonds of both countries. When instead arbitrage is segmented, arbitrageurs can invest in the currency of the home country (the riskless asset), and in a single additional asset class: foreign currency for some arbitrageurs, home bonds for others, and foreign bonds financed by borrowing in foreign currency for the remainder. We use segmented arbitrage as a benchmark for our analysis of global arbitrage.

In the case of global arbitrage, we denote by W_t the arbitrageurs' time-t wealth, by W_{Ft} their position in foreign assets and by $X_{Ht}^{(\tau)} d\tau$ and $X_{Ft}^{(\tau)} d\tau$ their position in the home and foreign bonds with maturities in $[\tau, \tau + d\tau]$, respectively, all expressed in units of the home currency. The arbitrageurs' budget constraint is

$$dW_t = W_t i_{Ht} dt + W_{Ft} \left(\frac{de_t}{e_t} + (i_{Ft} - i_{Ht}) dt \right)$$

$$+\int_0^T X_{Ht}^{(\tau)}\left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}}-i_{Ht}dt\right)d\tau+\int_0^T X_{Ft}^{(\tau)}\left(\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t}-\frac{de_t}{e_t}-i_{Ft}dt\right)d\tau. \tag{2.2}$$

If arbitrageurs invest all their wealth in the home short rate, then the instantaneous change dW_t in their wealth is $W_t i_{Ht} dt$, the first term in (2.2). Arbitrageurs can earn incremental returns relative to that case by investing in foreign currency, home bonds, and foreign bonds. The currency return derives from the CCT, which borrows short term in the home country and invests short term in the foreign country. The CCT's return corresponds to the second term in (2.2) and is $\frac{de_t}{e_t}+(i_{Ft}-i_{Ht})dt$, equal to the return on foreign currency plus that on the foreign-home short-rate differential. The home bond return derives from the home BCT, which borrows short-term in the home country and invests in home bonds. The home BCT's return corresponds to the third term in (2.2) and is $\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}}-i_{Ht}dt$, equal to the return on the home bond with maturity τ minus that on the home short rate. The foreign bond return is incremental relative to investing in foreign currency. It derives from the foreign BCT, which borrows short-term in the foreign country and invests in foreign bonds. The foreign BCT's return corresponds to the fourth term in (2.2) and is $\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t}-\frac{de_t}{e_t}-i_{Ft}dt$, equal to the return on the foreign bond with maturity τ minus that on the foreign short rate, with both returns expressed in home-currency terms.

The optimization problem of a global arbitrageur is

$$\max_{W_{Ft},\{X_{jt}^{(\tau)}\}_{\tau\in(0,T),j=H,F}}\left[\mathbb{E}_t(dW_t)-\frac{a}{2}\mathbb{V}\mathrm{ar}_t(dW_t)\right]. \tag{2.3}$$

The coefficient $a \geq 0$ characterizes the arbitrageur's risk aversion. It can capture in reduced form capital or Value-at-Risk constraints. By possibly redefining a, we assume that global arbitrageurs are in measure one.

In the case of segmented arbitrage, an arbitrageur's budget constraint is derived from (2.2) and optimization problem is derived from (2.3) by setting some of the investments to zero. For example, for an arbitrageur who can invest only in foreign currency we set the investments $X_{Ht}^{(\tau)}$ and $X_{Ft}^{(\tau)}$ to zero and optimize only over W_{Ft}. We denote by a_e, a_H, and a_F, respectively, the risk-aversion coefficient of an arbitrageur who can invest in foreign currency, home bonds, and foreign bonds. By possibly redefining (a_e, a_H, a_F), we assume that each type of arbitrageur is in measure one.

Currency traders generate a downward-sloping demand for foreign assets as a function of the exchange rate e_t, as in Hau and Rey (2006). We assume that this demand, expressed in units of the home currency, is affine and decreasing in the logarithm of the exchange rate:

$$Z_{et}=-\alpha_e\log(e_t)-(\zeta_{et}+\theta_e\gamma_t), \tag{2.4}$$

where $\alpha_e \geq 0$ is a slope coefficient that remains constant over time t, and $\zeta_{et} + \theta_e \gamma_t$ is an intercept coefficient. The intercept coefficient is the sum of a term ζ_{et} that depends deterministically on t and of a constant θ_e times a currency demand factor γ_t that depends stochastically on t. We describe the dynamics of ζ_{et} and γ_t later in this section. With a slight abuse of language, we refer to α_e and $\zeta_{et} + \theta_e \gamma_t$ as demand slope and demand intercept, respectively.

Currency traders can be interpreted as households or as central banks. For example, when e_t is low, the flow demand for foreign goods arising from home and foreign households may increase because of expenditure switching, and this may push up the net worth of foreign households, which is invested in foreign assets as in Gabaix and Maggiori (2015). Alternatively, when e_t is low, the central bank in the home country may want to increase its stock of foreign currency, perhaps to stabilize the currency. In ongoing work (Gourinchas, Ray, and Vayanos (2024, GRV1)), we endogenize the demand of currency traders in a two-country New Keynesian model in which currency and bond markets are segmented for households and are partly integrated by arbitrageurs with limited capital. The currency traders in GRV1 are the households and their demand is downward-sloping because of expenditure switching. The endogenous demand of currency traders in GRV1 incorporates two mechanisms not present in (2.4). First, changes to e_t affect the trade balance and hence the change in foreign assets, but move the stock of foreign assets only gradually. Second, monetary policy redistributes wealth across countries because it generates capital gains or losses for arbitrageurs which are distributed back to households. We assume away these mechanisms in (2.4) for tractability. These mechanisms can affect the response of the CCT and BCTs to monetary policy, working alongside the mechanisms that we derive in this paper.

Bond investors form clienteles, each of which has a preference ("habitat") for a bond of a specific country and maturity. Examples of clienteles are pension funds and insurance companies, which prefer long-maturity bonds in their country because these match their pension liabilities, which are long term and denominated in their country's currency. We denote the demand of bond investors for bonds of country j with maturities in $[\tau, \tau + d\tau]$, expressed in units of the home currency, by $Z_{jt}^{(\tau)} d\tau$. Following Vayanos and Vila (2021), we assume that $Z_{jt}^{(\tau)}$ is affine and decreasing in the logarithm of the bond price:

$$Z_{jt}^{(\tau)} = -\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) - \beta_{jt}^{(\tau)}, \tag{2.5}$$

where $\alpha_j(\tau) \geq 0$ is a slope coefficient that remains constant over time t but can depend on country j and maturity τ, and $\beta_{jt}^{(\tau)}$ is an intercept coefficient. The intercept coefficient is

$$\beta_{jt}^{(\tau)} = \zeta_j(\tau) + \theta_j(\tau)\beta_{jt}, \tag{2.6}$$

where $(\zeta_j(\tau), \theta_j(\tau))$ are independent of t but can depend on j and τ, and β_{jt} is a bond demand factor in country j that depends stochastically on t, depends on j and is independent of τ. With a slight abuse of language, we refer to $\alpha_j(\tau)$ and $\beta_{jt}^{(\tau)}$ as demand slope and demand intercept, respectively.

The 5×1 vector $q_t \equiv (i_{Ht}, i_{Ft}, \gamma_t, \beta_{Ht}, \beta_{Ft})^\top$ follows the process

$$dq_t = \Gamma(\bar{q} - q_t)dt + \Sigma dB_t, \tag{2.7}$$

where $\bar{q}$ is a constant 5×1 vector, (Γ, Σ) are constant 5×5 matrices, B_t is a 5×1 vector $(B_{iHt}, B_{iFt}, B_{\gamma t}, B_{\beta Ht}, B_{\beta Ft})^\top$ of independent Brownian motions, and $\top$ denotes transpose. Equation (2.7) nests the case where the factors $(i_{Ht}, i_{Ft}, \gamma_t, \beta_{Ht}, \beta_{Ft})$ are mutually independent, and the case where they are correlated. Independence arises when the matrices (Γ, Σ) are diagonal. When instead Σ is non-diagonal, shocks to the factors are correlated, and when Γ is non-diagonal, the drift (instantaneous expected change) of each factor depends on all other factors. We assume that the eigenvalues of Γ have positive real parts so that q_t is stationary. Equation (2.7) implies that the long-run mean of a stationary q_t is $\bar{q}$. We set the long-run means of the demand factors to zero ($\bar{q}_3 = \bar{q}_4 = \bar{q}_5 = 0$). This is without loss of generality because we can redefine ζ_{et} and $\{\zeta_j(\tau)\}_{j=H,F}$ to include a non-zero long-run mean. We set the supply of foreign assets and of home and foreign bonds to zero by redefining investor demand to be net of supply.

We express all demand functions in units of the home currency, which we assume is the riskless asset for arbitrageurs, for tractability. Expressing some demand functions in the foreign currency renders our model nonlinear, but its linearized version remains the same.

While we model the demand for foreign assets, home bonds, and foreign bonds as corresponding to different risk factors, these demands can be correlated. For example, suppose that some currency traders trade foreign-currency forwards and swaps. We show in Appendix A that under global arbitrage, Covered Interest Parity (CIP) holds, and an increase in the demand for foreign-currency forwards and swaps translates into an equal increase in the demand for foreign assets and foreign bonds, and an equal decrease in the demand for home bonds. Our model can capture correlated demands through the matrices Γ and Σ. From now on, we use the terms foreign assets and foreign currency interchangeably.

Our model can be given both a nominal and a real interpretation. Under the nominal interpretation, emphasized so far, the exchange rate is the price of one currency relative to the other and bonds pay in currency units. Under the real interpretation, the exchange rate e_t is the real exchange rate defined as the price of goods in one country relative to the other and bonds pay in units of goods.

A difficulty with the nominal interpretation is that the demand of currency traders is better viewed as a function of the real, rather than the nominal, exchange rate. To make the nominal interpretation compatible with a real currency demand, we can replace the nominal exchange rate e_t in (3.1) by the real exchange rate. This amounts to keeping e_t inside the logarithm and adding $\alpha_e(\log(p_{Ft}) - \log(p_{Ht}))$ to ζ_{et}, where p_{jt} is the price level in country $j = H, F$. In subsequent sections we present the nominal interpretation of our model allowing for a real currency demand and restricting inflation to be constant in each country: $\zeta_{et} = \zeta_e + \alpha_e(\pi_F - \pi_H)t$, where π_j is the constant inflation rate in country j and ζ_e is a constant. In GRV1 we extend our analysis so that inflation is stochastic and determined endogenously within a New Keynesian model.

3 Equilibrium

In this section we characterize the equilibrium with global arbitrage. We start by conjecturing that the equilibrium exchange rate and bond yields are log-affine functions of q_t. That is, there exist six scalars $(\{A_{ije}\}_{j=H,F}, A_{\gamma e}, \{A_{\beta je}\}_{j=H,F}, C_e)$ and twelve functions $(\{A_{ijj'}(\tau)\}_{j,j'=H,F}, \{A_{\gamma j}(\tau)\}_{j=H,F}, \{A_{\beta jj'}(\tau)\}_{j,j'=H,F}, \{C_j(\tau)\}_{j=H,F})$ that depend only on τ, such that

$$\log e_t = -\left[A_e^\top q_t + C_e + (\pi_F - \pi_H)t\right], \tag{3.1}$$

$$\log P_{jt}^{(\tau)} = -\left[A_j(\tau)^\top q_t + C_j(\tau)\right], \tag{3.2}$$

where $A_e \equiv (A_{iHe}, -A_{iFe}, A_{\gamma e}, A_{\beta He}, -A_{\beta Fe})^\top$ and $A_j(\tau) \equiv (A_{iHj}(\tau), A_{iFj}(\tau), A_{\gamma j}(\tau), A_{\beta Hj}(\tau), A_{\beta Fj}(\tau))^\top$. Applying Ito's Lemma to (3.1) and using (2.7), we find

$$\frac{de_t}{e_t} = \mu_{et}dt - A_e^\top \Sigma dB_t, \tag{3.3}$$

where

$$\mu_{et} \equiv A_e^\top \Gamma(q_t - \bar{q}) - (\pi_F - \pi_H) + \frac{1}{2}A_e^\top \Sigma\Sigma^\top A_e. \tag{3.4}$$

Likewise, applying Ito's Lemma to (3.2) for $j = H, F$, using (2.7), and additionally using (3.1) in the case $j = F$, we find

$$\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \mu_{Ht}^{(\tau)}dt - A_H(\tau)^\top \Sigma dB_t, \tag{3.5}$$

$$\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} = \mu_{Ft}^{(\tau)}dt - A_F(\tau)^\top \Sigma dB_t, \tag{3.6}$$

where

$$\mu_{Ht}^{(\tau)} \equiv A'_H(\tau)^\top q_t + C'_H(\tau) + A_H(\tau)^\top \Gamma(q_t - \bar{q}) + \frac{1}{2} A_H(\tau)^\top \Sigma\Sigma^\top A_H(\tau), \tag{3.7}$$

$$\mu_{Ft}^{(\tau)} \equiv A'_F(\tau)^\top q_t + C'_F(\tau) + A_F(\tau)^\top \Gamma(q_t - \bar{q}) + \frac{1}{2} A_F(\tau)^\top \Sigma\Sigma^\top \left(A_F(\tau) + 2A_e\right). \tag{3.8}$$

We next derive the arbitrageurs' first-order conditions. Substituting (3.3), (3.5) and (3.6) into the budget constraint (2.2), we can write the optimization problem (2.3) as

$$\max_{W_{Ft}, \{X_{jt}^{(\tau)}\}_{\tau \in (0,T), j=H,F}} \Bigg[W_{Ft}\left(\mu_{et} + i_{Ft} - i_{Ht}\right) + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} \left(\mu_{jt}^{(\tau)} - i_{jt}\right) d\tau$$
$$-\frac{a}{2} \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right)^\top \Sigma\Sigma^\top \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right) \Bigg]. \tag{3.9}$$

The first-order condition with respect to W_{Ft} is

$$\mu_{et} + i_{Ft} - i_{Ht} = A_e^\top \lambda_t, \tag{3.10}$$

and the first-order condition with respect to $X_{jt}^{(\tau)}$ is

$$\mu_{jt}^{(\tau)} - i_{jt} = A_j(\tau)^\top \lambda_t, \tag{3.11}$$

where $j = H, F$ and the vector $\lambda_t \equiv (\lambda_{iHt}, \lambda_{iFt}, \lambda_{\beta Ht}, \lambda_{\beta Ft}, \lambda_{\gamma t})^\top$ of the prices of the risk factors is

$$\lambda_t = a\Sigma\Sigma^\top \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right). \tag{3.12}$$

The first-order condition (3.10) describes the arbitrageurs' risk-return trade-off when investing in the CCT. If arbitrageurs invest an extra unit of home currency in the CCT, then their expected return increases by $\mu_{et} + i_{Ft} - i_{Ht}$, the left-hand side of (3.10). The right-hand side of (3.10) is the increase in the arbitrageurs' portfolio risk, times their risk-aversion coefficient a. The increase in portfolio risk is equal to the covariance between the return on the CCT and the return on the arbitrageurs' portfolio. That covariance is the product of the vectors that describe the sensitivity of each return to the risk factors, times the factors' covariance matrix $\Sigma\Sigma^\top$. The sensitivity of the CCT's return to the factors is A_e, and the sensitivity of the portfolio return is $W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau$.

The first-order condition (3.11) describes the arbitrageurs' risk-return trade-off when investing in the BCT in country j. If arbitrageurs invest an extra unit of home currency in the BCT for country j and maturity τ, then their expected return increases by $\mu_{jt}^{(\tau)} - i_{jt}$, the left-hand side of (3.11). The right-hand side of (3.11) is the increase in the arbitrageurs' portfolio risk, times their risk-aversion coefficient a. The increase in portfolio risk is equal to the covariance between the return on the BCT in country j and for maturity τ, and the return on the arbitrageurs' portfolio.

We next combine the first-order conditions with market clearing. Clearing in the currency market requires

$$W_{Ft} + Z_{et} = 0. \tag{3.13}$$

Clearing in the market for country j bonds with maturity τ requires

$$X_{jt}^{(\tau)} + Z_{jt}^{(\tau)} = 0. \tag{3.14}$$

Using (3.13) and (3.14), we can write λ_t as

$$\lambda_t = a\Sigma\Sigma^\top \left(-Z_{et}A_e - \sum_{j=H,F} \int_0^T Z_{jt}^{(\tau)} A_j(\tau) d\tau \right). \tag{3.15}$$

Substituting the demands Z_{et} and $\{Z_{jt}^{(\tau)}\}_{j=H,F}$ from (2.4) and (2.5), respectively, and using (3.1) and (3.2), we can write (3.15) as

$$\begin{aligned}\lambda_t = a\Sigma\Sigma^\top \Bigg(&\left(\zeta_e + \theta_e \gamma_t - \alpha_e \left(A_e^\top q_t + C_e\right)\right) A_e \\ &+ \sum_{j=H,F} \int_0^T \left(\zeta_j(\tau) + \theta_j(\tau)\beta_{jt} - \alpha_j(\tau)\left(A_j(\tau)^\top q_t + C_j(\tau)\right)\right) A_j(\tau) d\tau \Bigg).\end{aligned} \tag{3.16}$$

Substituting λ_t from (3.16) and μ_{et} from (3.4) into (3.10), we find an equation that is affine in q_t. We find two additional affine equations by substituting λ_t from (3.16), $\mu_{Ht}^{(\tau)}$ from (3.7) and $\mu_{Ft}^{(\tau)}$ from (3.8) into (3.10) for $j = H, F$. Identifying linear and constant terms yields a system of scalar equations and ordinary differential equations (ODEs). That system reduces to one of 25 nonlinear scalar equations, as shown in Proposition B.1. Proposition B.1 is stated and proved in Appendix B, which contains the proofs of all results in Sections 3 and 4.

Our model yields Uncovered Interest Parity (UIP) and the Expectations Hypothesis (EH) as a special case. Both properties hold when arbitrageurs are risk-neutral ($a = 0$). Setting $a = 0$ in (3.10) yields $\mu_{et} = i_{Ht} - i_{Ft}$, as under UIP. Setting $a = 0$ in (3.11) yields $\mu_{jt}^{(\tau)} \equiv i_{jt}$, as under EH. An additional equation that must hold when $a = 0$, as shown in Proposition B.1, is

$$\bar{i}_H - \pi_H = \bar{i}_F - \pi_F + \frac{1}{2}\left(\mathcal{E}_{iH} - \mathcal{E}_{iF}\right)^\top \Gamma^{-1}\Sigma\Sigma^\top \left(\Gamma^{-1}\right)^\top \left(\mathcal{E}_{iH} - \mathcal{E}_{iF}\right), \tag{3.17}$$

where $\bar{i}_j$ denotes the unconditional mean of the nominal short rate in country j and $(\mathcal{E}_{iH}, \mathcal{E}_{iF})$ denote the first two columns of the 5×5 identity matrix. According to (3.17), the unconditional mean $\bar{i}_j - \pi_j$ of the real interest rate in country j must be equal across home and foreign, up to a convexity adjustment (the last term in (3.17)). Equality of average real rates is a restriction on model parameters. It must hold for $a = 0$ because of the stationarity of the real exchange rate, which is implicit in the conjectured form (3.1). Indeed, if average real rates differed across countries and arbitrageurs were risk-neutral, then the real exchange rate would appreciate on average or depreciate on average forever, violating stationarity.

When arbitrageurs are risk-averse, a stationary equilibrium can exist even when average real rates differ across countries and even in the limit when risk aversion goes to zero. This is because any difference in average real rates is absorbed in equilibrium by an adjustment in currency risk premia. The currency of the country with a higher average real rate is permanently stronger and earns a positive premium. Arbitrageurs have a high demand for that currency, through their position in the CCT, and earn the positive premium. Currency traders have offsetting low demand if their demand is price-elastic ($\alpha_e > 0$). In the limit when arbitrageurs' risk aversion a goes to zero, risk premia remain non-zero and arbitrageurs' position in the CCT becomes arbitrarily large. Corollary B.1 in Appendix B summarizes these results.

4 No Demand Risk

In this section we study the case where the demand for foreign currency, home bonds, and foreign bonds does not vary stochastically: the demand factors $(\gamma_t, \beta_{Ht}, \beta_{Ft})$ have zero variance and are equal to their long-run mean of zero. For simplicity we also assume that the home and foreign short rates (i_{Ht}, i_{Ft}) are independent and that one-off shocks to the demand factors do not affect the short rates or other demand factors. Our assumptions amount to taking the matrices (Γ, Σ) in (2.7) to be diagonal and to setting $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. Setting $(\Gamma_{1,1}, \Gamma_{2,2}, \bar{q}_1, \bar{q}_2, \Sigma_{1,1}, \Sigma_{2,2}) \equiv (\kappa_{iH}, \kappa_{iF}, \bar{i}_H, \bar{i}_F, \sigma_{iH}, \sigma_{iF})$, we can write the dynamics of the country-j short rate as

$$di_{jt} = \kappa_{ij}(\bar{i}_j - i_{jt})dt + \sigma_{ij} dB_{ijt}. \tag{4.1}$$

The simplifying assumptions in this section allow us to study the equilibrium analytically.

4.1 Segmented Arbitrage

When arbitrage is segmented, the first-order condition of the arbitrageurs in each market reflects their own risk aversion and portfolio composition. Because arbitrageurs in currency and bond markets hold different portfolios, the prices of the risk factors differ across these markets. Solving for equilibrium in the currency market reduces to solving a nonlinear scalar equation, as shown in Proposition B.2 in Appendix B. The same is true for equilibrium in the country-j bond market, as shown in Proposition B.3 in Appendix B. Using our characterization of equilibrium, we next derive equilibrium properties.

4.1.1 Short-Rate Shocks and Carry Trades

Proposition 4.1. *Suppose that arbitrage is segmented. Following a drop in the home short rate or a rise in the foreign short rate, the foreign currency appreciates ($A_{iHe} > 0$, $A_{iFe} > 0$). When additionally currency arbitrageurs are risk-averse ($a_e > 0$) and the demand of currency traders is price-elastic ($\alpha_e > 0$),*

- *The foreign currency does not appreciate all the way to the level implied by UIP: $A_{iHe} < A_{iHe}^{UIP} \equiv \frac{1}{\kappa_{iH}}$, $A_{iFe} < A_{iFe}^{UIP} \equiv \frac{1}{\kappa_{iF}}$.*
- *The expected return of the CCT rises: $\frac{\partial(\mu_{et}+i_{Ft}-i_{Ht})}{\partial i_{Ht}} < 0$ and $\frac{\partial(\mu_{et}+i_{Ft}-i_{Ht})}{\partial i_{Ft}} > 0$.*

When the home short rate drops or the foreign short rate rises, the foreign currency appreciates. These movements are in the direction implied by UIP. The foreign currency does not appreciate all the way to the value implied by UIP, however. Intuitively, a drop in i_{Ht} or a rise in i_{Ft} render the CCT more profitable, raising its expected return $\mu_{et} + i_{Ft} - i_{Ht}$ and inciting currency arbitrageurs to increase their holdings W_{Ft} of the foreign currency. This puts upward pressure on the exchange rate. When the demand of currency traders is price-elastic, their holdings Z_{et} decrease as the foreign currency appreciates and those of currency arbitrageurs W_{Ft} increase in equilibrium. Risk-averse arbitrageurs, however, do not trade all the way to the point where e_t reaches its UIP value. Instead, in a spirit similar to Gabaix and Maggiori (2015), the CCT's expected return $\mu_{et} + i_{Ft} - i_{Ht}$ remains higher than before the shock to compensate arbitrageurs for the risk generated by their larger foreign-currency position. The exchange rate adjusts all the way to its UIP value when currency arbitrageurs are risk-neutral or when the demand of currency traders is price-inelastic. In the latter case this is because arbitrageurs' activity causes prices to rise up to the point where there is no change in W_{Ft}.

Proposition 4.1 implies that the difference between the foreign and the home short rate predicts positively the CCT's future return. This is consistent with the findings of Bilson (1981) and Fama (1984) that following an increase in the foreign-minus-home short-rate differential, the expected return on the foreign currency typically increases.

Proposition 4.2. *Suppose that arbitrage is segmented. Following a drop in the short rate in country j, bond yields drop in that country ($A_{ijj}(\tau) > 0$) and do not change in the other country ($A_{ij'j}(\tau) = 0$ for $j' \neq j$). When additionally bond arbitrageurs in country j are risk-averse ($a_j > 0$) and the demand of bond investors in that country is price-elastic ($\alpha_j(\tau) > 0$):*

- *Bond yields do not drop all the way to the value implied by the EH:* $A_{ijj}(\tau) < A_{ijj}^{EH}(\tau) \equiv \frac{1-e^{-\kappa_{ij}\tau}}{\kappa_{ij}}$.
- *The expected return of the BCT rises:* $\frac{\partial\left(\mu_{jt}^{(\tau)}-i_{jt}\right)}{\partial i_{jt}} < 0$.

When the short rate in country j drops, bond prices in that country rise (and bond yields drop) because of a standard discounting effect. Prices do not rise all the way to the value implied by the EH, however. Intuitively, a drop in the short rate renders the BCT in country j more profitable, raising its expected return $\mu_{jt}^{(\tau)} - i_{jt}$ and inciting bond arbitrageurs in country j to increase their bond holdings $X_{jt}^{(\tau)}$. This puts upward pressure on bond prices $P_{jt}^{(\tau)}$. When the demand of bond investors in country j is price-elastic, their holdings $Z_{jt}^{(\tau)}$ decrease as bond prices rise and those of bond arbitrageurs $X_{jt}^{(\tau)}$ increase in equilibrium. Risk-averse arbitrageurs, however, do not trade all the way to the point where bond prices reach their EH value. Instead, as Vayanos and Vila (2021) show for a closed economy, the BCT's expected return $\mu_{jt}^{(\tau)} - i_{jt}$ remains higher than before the shock to compensate arbitrageurs for the risk generated by their larger bond position. Bond prices adjust all the way to their EH value when bond arbitrageurs in country j are risk-neutral or when the demand of bond investors in country j is price-inelastic.

Proposition 4.2 implies that the slope of the term structure in country j predicts positively the BCT's future return in that country. This is because both slope and expected return are high when the country j short rate i_{jt} is low. A positive relationship between term-structure slope and BCT expected return is documented in Fama and Bliss (1987).

4.1.2 Demand Shocks

We next determine how the exchange rate and bond yields respond to changes in the demand for foreign currency, home bonds, and foreign bonds. Since we assume no demand risk in this section, we take the demand changes to be unanticipated and one-off. Demand changes by currency traders correspond to shocks to the currency demand factor γ_t. Demand changes by bond investors in country j correspond to shocks to the country j bond demand factor β_{jt}. Following the shocks, the demand factors revert deterministically to their mean of zero.

Without loss of generality, we take θ_e to be positive, which means that an increase in γ_t corresponds to a drop in demand for foreign currency. We take $\theta_j(\tau)$ to be positive for all τ, which means that an increase in β_{jt} corresponds to a drop in demand for the bonds of country j.

Proposition 4.3. *Suppose that arbitrage is segmented, $\theta_e > 0$ and $\theta_j(\tau) > 0$ for all τ.*

- *An unanticipated one-off drop in currency traders' demand for foreign currency (increase in γ_t) causes the foreign currency to depreciate if currency arbitrageurs are risk-averse ($a_e > 0$). It has no effect on bond yields.*

- *An unanticipated one-off drop in investor demand for the bonds of country j (increase in β_{jt}) raises bond yields in country j if bond arbitrageurs in that country are risk-averse ($a_j > 0$). It has no effect on bond yields in the other country and on the exchange rate.*

When arbitrage is segmented, changes to the demand for an asset class—foreign currency, home bonds, foreign bonds—affect that asset class only. When, for example, the demand for bonds in country j drops, these bonds become cheaper and their yields increase, while foreign currency and bonds in the other country are unaffected.

4.2 Global Arbitrage

When arbitrage is global, the prices of the risk factors are equal across currency and bond markets. Solving for equilibrium reduces to solving a system of three nonlinear scalar equations, as shown in Proposition B.4 in Appendix B. Using our characterization of equilibrium, we next derive equilibrium properties.

4.2.1 Short-Rate Shocks and Carry Trades

Proposition 4.4. *Suppose that arbitrage is global.*

- *The effects of shocks to the country-j short rate i_{jt} on the exchange rate and the CCT's expected return have the same properties as in Proposition 4.1, and on bond yields in country j and the BCT's expected return have the same properties as in Proposition 4.2, except that the price-elasticity condition can hold for currency traders or bond investors ($\alpha_e > 0$ or $\alpha_j(\tau) > 0$).*
- *When arbitrageurs are risk-averse ($a > 0$) and the demand of currency traders is price-elastic ($\alpha_e > 0$), a drop in i_{jt} causes bond yields in country $j' \neq j$ to drop ($A_{j'j}(\tau) > 0$) and the BCT's expected return to drop ($\frac{\partial\left(\mu^{(\tau)}_{j't} - i_{j't}\right)}{\partial i_{jt}} > 0$).*
- *The effect of i_{jt} on bond yields is smaller in country j' than in country j ($A_{jj}(\tau) > A_{j'j}(\tau)$).*

Bond yields respond to shocks differently under global and segmented arbitrage. Under segmented arbitrage, a shock to the short rate i_{jt} in country j affects bond yields in that country only. By contrast, under global arbitrage, and provided that $a\alpha_e > 0$, the shock affects bond yields in both countries, even though the short rate $i_{j't}$ in country $j' \neq j$ does not change. When i_{jt} drops, bond yields in both countries drop.

Short-rate shocks are transmitted across countries because global arbitrageurs engage in the CCT and use the bond market to hedge. Recall that under both segmented and global arbitrage, a drop in the home short rate i_{Ht} raises the profitability of the CCT, making it more attractive to arbitrageurs. When the demand of currency traders is price-elastic, the arbitrageurs' equilibrium investment in the CCT increases. Because arbitrageurs hold more foreign currency (higher W_{Ft}), they become more exposed to the risk that the foreign short rate i_{Ft} drops and the foreign currency depreciates. Global arbitrageurs hedge that risk by buying foreign bonds because their price rises when i_{Ft} drops. This pushes the prices of foreign bonds up and their yields down.

An additional consequence of hedging by global arbitrageurs is greater under-reaction of the exchange rate and bond yields to short rates. When i_{Ht} drops, arbitrageurs invest more in the CCT and in the home BCT. Each of these trades exposes them to a rise in i_{Ht}. Hence, global arbitrageurs are less eager than segmented arbitrageurs to buy foreign currency and home bonds following a drop in i_{Ht}, and the expected return of the CCT and the home BCT increase more than under segmented arbitrage. In particular, when the demand of currency traders is inelastic and that by bond investors is elastic, a drop in i_{Ht} raises the CCT's expected return under global arbitrage but leaves it unaffected under segmented arbitrage. Likewise, when the demand of home bond investors is inelastic and that by currency traders is elastic, a drop in i_{Ht} raises the home BCT's expected return under global arbitrage but leaves it unaffected under segmented arbitrage.

We next turn to variants of the CCT studied in the empirical literature. One variant is a hybrid CCT in which the trading horizon is short but the trading instruments are long-term. Borrowing in the home country and investing in the foreign country is done with the respective τ-year bonds, and the positions are held for a short horizon dt. The return of the hybrid CCT in home-currency units is

$$\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \left(\frac{de_t}{e_t} + (i_{Ft} - i_{Ht})dt\right) + \left(\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - i_{Ft}dt\right) - \left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht}dt\right). \tag{4.2}$$

Hence, the hybrid CCT can be viewed as a combination of (i) the basic CCT, (ii) a long position in the foreign BCT, and (iii) a short position in the home BCT.

A second variant is a long-horizon CCT, in which borrowing in the home country and investing in the foreign country is done with the respective τ-year bonds, and the positions are held until the bonds' maturity. The return of the long-horizon CCT in home-currency units and log terms is

$$\log\left(\frac{e_{t+\tau}}{P_{Ft}^{(\tau)}e_t}\right) - \log\left(\frac{1}{P_{Ht}^{(\tau)}}\right) = \int_t^{t+\tau}\left(\log\left(\frac{e_{s+ds}}{e_s}\right) + i_{Fs}ds - i_{Hs}ds\right) + \left(\tau y_{Ft}^{(\tau)} - \int_t^{t+\tau} i_{Fs}ds\right) - \left(\tau y_{Ht}^{(\tau)} - \int_t^{t+\tau} i_{Hs}ds\right), \tag{4.3}$$

where the equality follows from (2.1). Hence, the long-horizon CCT can be viewed as the combination of (i) a sequence of basic CCTs, (ii) a long position in a long-horizon foreign BCT, and (iii) a short position in a long-horizon home BCT. The long-horizon BCT in country j involves buying bonds in country j and financing that position by borrowing short-term and rolling over. Proposition 4.5 characterizes the expected returns of the hybrid CCT and the long-horizon CCT. We express the expected return of the long-horizon CCT on an annualized basis, as we do for all other expected returns.

Proposition 4.5. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), and the demand of currency traders or of bond investors is price-elastic ($\alpha_e > 0$ or $\alpha_j(\tau) > 0$).*

- *The hybrid CCT's and the long-horizon CCT's expected returns rise following a drop in the home short rate i_{Ht} or a rise in the foreign short rate i_{Ft}, provided that the maturity τ of the bonds involved in these trades lies in an interval $(0, \tau^*)$. The threshold τ^* is infinite when countries are symmetric.*
- *The sensitivity of the hybrid CCT's expected return to (i_{Ht}, i_{Ft}) is smaller than for the basic CCT. The sensitivity of the long-horizon CCT's expected return to (i_{Ht}, i_{Ft}) is smaller than for the corresponding sequence of basic CCTs.*
- *When maturity τ goes to infinity, regardless of whether $(a, \alpha_e, \alpha(\tau))$ are non-zero:*
  - *The expected returns of the hybrid CCT and long-horizon CCT go to zero.*
  - *The difference in real yields across countries goes to zero.*

Short-rate shocks move the expected returns of the hybrid CCT and the long-horizon CCT in the same direction as for the basic CCT, except possibly when the maturity τ of the bonds involved in these trades is long. The effects of short-rate shocks on the hybrid CCT and the long-horizon CCT are smaller than for the corresponding basic CCTs because the shocks' effects through the BCTs work in the opposite direction. Consider, for example, a drop in the home short rate. Proposition 4.4 implies that the expected return of the basic CCT increases, but so does the expected return of the home BCT, which enters as a short position in the hybrid CCT and the long-horizon CCT.

When the maturity τ of the bonds involved in the hybrid CCT and the long-horizon CCT goes to infinity, the effects of short-rate shocks through the BCTs offset fully those through the basic CCT. As a consequence, short-rate shocks have no effect on the hybrid CCT's and the long-horizon CCT's expected returns. The expected returns of both trades go to zero, and so does the difference in real yields across countries. The proofs of these convergence results require that the functions $\{A_j(\tau)\}_{j=H,F}$ converge to finite limits, or equivalently that the eigenvalues of the matrix M have positive real parts. These conditions hold in the absence of demand risk, as shown in Lemma B.1 in Appendix B, but do not necessarily hold with demand risk. They do not hold, in particular, in our estimated model in Section 5.

4.2.2 Demand Shocks

Under global arbitrage, shocks to the demand for an asset class (foreign currency, home bonds, foreign bonds) affect all other assets. This is in contrast to segmented arbitrage, where only the asset class for which demand changes is affected (Proposition 4.3).

Proposition 4.6. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), the functions $(\alpha_H(\tau), \alpha_F(\tau))$ are non-increasing, and $\theta_e > 0$. An unanticipated one-off drop in currency traders' demand for foreign currency (increase in γ_t):*

- *Causes the foreign currency to depreciate.*
- *Raises bond yields in the home country.*
- *Lowers bond yields in the foreign country.*

A drop in currency traders' demand for foreign currency causes it to depreciate, as in Proposition 4.3. Additionally, hedging by global arbitrageurs causes home bond prices to drop and foreign bond prices to rise. Indeed, arbitrageurs accommodate the drop in demand for foreign currency by holding more of it. Hence, they become more exposed to a rise in the home short rate i_{Ht} and to a decline in the foreign short rate i_{Ft}. This makes them less willing to hold home bonds, which drop in price when i_{Ht} rises, and more willing to hold foreign bonds, which rise in price when i_{Ft} drops.

Proposition 4.7. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), the functions $(\alpha_H(\tau), \alpha_F(\tau))$ are non-increasing, and the function $\theta_j(\tau)$ is positive. An unanticipated one-off drop in investor demand for the bonds of country j (increase in β_{jt}):*

- *Raises bond yields in country j.*
- *Raises bond yields in country $j' \neq j$ when the demand of currency traders is price-elastic ($\alpha_e > 0$).*
- *Causes the foreign currency to depreciate if $j = H$, and to appreciate if $j = F$.*

A drop in investor demand for home bonds depresses their prices, as in Proposition 4.3. Additionally, hedging by global arbitrageurs causes prices for foreign bonds to drop and the foreign currency to depreciate. Indeed, arbitrageurs accommodate the drop in demand for home bonds by holding more such bonds. Hence, they become more exposed to a rise in the home short rate i_{Ht}. This makes them less willing to hold foreign currency, which depreciates when i_{Ht} rises. If the demand of currency traders is price-elastic, then arbitrageurs hold less foreign currency in equilibrium. Hence, they become less exposed to a drop in the foreign short rate i_{Ft} and less willing to hold foreign bonds, which rise in price when i_{Ft} drops. A drop in demand for foreign bonds has symmetric effects.

4.3 Monetary Policy Transmission

Our analysis has implications for the domestic and international transmission of monetary policy. Consider first conventional monetary easing at home, modelled as an unanticipated cut to the home short rate i_{Ht} (Propositions 4.1, 4.2 and 4.4). The rate cut propagates imperfectly along the home term structure and causes the home currency to depreciate. Assuming that the foreign short rate remains unchanged, yields on foreign bonds do not move under segmented arbitrage. Under global arbitrage instead, foreign bond yields decrease. Thus, there is imperfect insulation under global arbitrage, in the sense that foreign monetary conditions, interpreted as the shape of entire yield curve, are affected by domestic monetary conditions.

Consider next QE purchases of home bonds, modelled as an unanticipated drop in the demand factor β_{Ht} (Propositions 4.3 and 4.7). This policy decreases home bond yields. The exchange rate and yields of foreign bonds remain unchanged under segmented arbitrage. Under global arbitrage instead, yields on foreign bonds decrease and the home currency depreciates. Once again, foreign monetary conditions are affected by domestic monetary conditions under global arbitrage, and there is imperfect insulation.

For both conventional and unconventional policies, domestic and foreign monetary conditions co-move positively: easing at home eases abroad and vice versa. Moreover, both types of policies affect the exchange rate: conventional easing causes the home currency to depreciate, and QE does the same. The imperfect insulation result with global arbitrage is at odds with the Trilemma, provided that the latter is broadly interpreted as stating that with perfect capital mobility a floating exchange rate provides monetary autonomy not just in setting short rates but also in shaping the entire yield curve.[5]

5 Demand Risk

In this section we return to the full model analyzed in Section 3 with global arbitrage and stochastic demand for bonds and foreign currency. We estimate the model parameters from the data and use the estimated model to examine two sets of issues. First, the drivers of currency and bond returns, as well as the predictability of these returns. Second, the domestic and international transmission of monetary policy.

5.1 Estimation

Our baseline estimation strategy is to use Maximum Likelihood Estimation (MLE) on time-series data on exchange rates and bond yields. As an alternative estimation strategy,

[5]Monetary autonomy can be recovered if the central bank uses a combination of conventional and unconventional policy tools. We leave studying the design of such policies for future work. See Kamdar and Ray (2024) for a model of optimal short-rate and balance-sheet tools in a closed economy.

we use Generalized Method of Moments (GMM), matching model-implied moments on exchange rates, bond yields, and trading volume to their empirical counterparts. In Appendix C.4, we describe our GMM estimation method and results. The GMM point estimates are similar to the MLE ones but yield significantly wider confidence intervals for the exchange-rate results.

To compute the log-likelihood, we assume that observations are made once every Δt years. Integrating the dynamics (2.7) of the 5×1 vector $q_t \equiv (i_{Ht}, i_{Ft}, \gamma_t, \beta_{Ht}, \beta_{Ft})^\top$ between t and $t + \Delta t$, we find

$$q_{t+\Delta t} - \bar{q} = e^{-\Gamma \Delta t}(q_t - \bar{q}) + \int_t^{t+\Delta t} e^{-\Gamma(t+\Delta t - s)} \Sigma dB_s. \tag{5.1}$$

The covariance matrix $\hat{\Sigma}_{\Delta t} \equiv \int_t^{t+\Delta t} e^{-\Gamma(t+\Delta t-s)} \Sigma \Sigma^\top e^{-\Gamma^\top (t+\Delta t - s)} ds$ of the innovation term $\int_t^{t+\Delta t} e^{-\Gamma(t+\Delta t-s)} \Sigma dB_s$ satisfies the Lyapunov equation

$$\Gamma \hat{\Sigma}_{\Delta t} + \hat{\Sigma}_{\Delta t} \Gamma^\top = \Sigma \Sigma^\top - e^{-\Gamma \Delta t} \Sigma \Sigma^\top e^{-\Gamma^\top \Delta t}. \tag{5.2}$$

We observe a $K \times 1$ vector p_t of demeaned log exchange-rate and bond-yield data that are related to q_t through $p_t = A(q_t - \bar{q})$, where $K \geq 5$ and the $K \times 5$ matrix A is endogenously determined from the model parameters. Assuming that A has full rank (=5), we can write the dynamics of p_t as

$$p_{t+\Delta t} = A e^{-\Gamma \Delta t} A^+ p_t + A \int_t^{t+\Delta t} e^{-\Gamma(t+\Delta t-s)} \Sigma dB_s, \tag{5.3}$$

where $A^+ = (A^\top A)^{-1} A^\top$ is the Moore-Penrose inverse of A. Since the distribution of $p_{t+\Delta t}$ conditional on p_t is normal with mean $A e^{-\Gamma \Delta t} A^+ p_t$ and covariance matrix $A \hat{\Sigma}_{\Delta t} A^\top$, the log-likelihood is

$$\begin{aligned} \mathcal{L} \equiv & -\frac{1}{2} \sum_{t=1}^{\mathcal{T}} \left(p_{t+\Delta t} - A e^{-\Gamma \Delta t} A^+ p_t\right)^\top \left(A \hat{\Sigma}_{\Delta t} A^\top\right)^{-1} \left(p_{t+\Delta t} - A e^{-\Gamma \Delta t} A^+ p_t\right) \\ & - \frac{\mathcal{T}}{2} \log \det \left(A \hat{\Sigma}_{\Delta t} A^\top\right), \end{aligned} \tag{5.4}$$

where $\mathcal{T}$ is the length of the time-series. The estimated model parameters are the matrices Γ and Σ that determine the drift and diffusion, respectively, of q_t, the scalars (α_e, θ_e) that determine the demand of currency traders, the functions $\{(\alpha_j(\tau), \theta_j(\tau))\}_{j=H,F}$ that determine the demand of bond investors in each country, and the coefficient a of arbitrageur risk aversion. The parameters $(\zeta_e, \{\zeta_j(\tau)\}_{j=H,F})$ in the demand intercepts determine long-run means rather than responses to shocks, and we leave them out of our estimation that uses demeaned data p_t. We also leave out of our estimation the inflation rates because we assume that they are equal across countries ($\pi_H = \pi_F$) and thus that the exchange rate has no trend. Assuming no persistent differences in inflation rates across countries is a reasonable approximation in our estimation, where we take the home country to be the United States (US) and the foreign country to be the Eurozone (EZ).

We reduce the model parameters to a set that is manageable yet sufficiently rich. We take the mean-reversion matrix Γ to be diagonal except for the non-zero terms $\Gamma_{3,1}$ and $\Gamma_{3,2}$. Thus, the factors do not respond to each other's movements except for the currency demand factor γ_t that responds to short-rate movements in both countries. We take the covariance matrix Σ to be diagonal except for the non-zero term $\Sigma_{1,2}$ (setting $\Sigma_{2,1}$ to zero is without loss of generality because the data only identify $\Sigma\Sigma^\top$). Thus, innovations to the factors are independent except for those to short rates. We take the mean reversion of the demand factors and the standard deviation of innovations to these factors to be the same across home and foreign, setting $\Gamma_{4,4} = \Gamma_{5,5}$ and $\Sigma_{4,4} = \Sigma_{5,5}$. The restrictions on (Γ, Σ) simplify the estimation of the model and the interpretation of the results, while allowing us to capture two key features of the data: the correlation between short rates across countries, and the gradual response of exchange rates to short-rate shocks. Allowing innovations to the currency demand factor to correlate with innovations to the bond demand factors, as implied by our analysis of forwards and swaps in Appendix A, yields low correlations and similar estimates for the remaining parameters, as shown in Appendix C.2. With our assumed restrictions and analogous notation to that in Section 4, we can write Γ and Σ as

$$\Gamma = \begin{bmatrix} \kappa_{iH} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{iF} & 0 & 0 & 0 \\ \kappa_{\gamma,iH} & \kappa_{\gamma,iF} & \kappa_\gamma & 0 & 0 \\ 0 & 0 & 0 & \kappa_\beta & 0 \\ 0 & 0 & 0 & 0 & \kappa_\beta \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_{iH} & 0 & 0 & 0 & 0 \\ \sigma_{iH,iF} & \sigma_{iF} & 0 & 0 & 0 \\ 0 & 0 & \sigma_\gamma & 0 & 0 \\ 0 & 0 & 0 & \sigma_\beta & 0 \\ 0 & 0 & 0 & 0 & \sigma_\beta \end{bmatrix}. \tag{5.5}$$

We assume that the bond demand functions $\{(\alpha_j(\tau), \theta_j(\tau))\}_{j=H,F}$ take the exponential form

$$\alpha_j(\tau) = \alpha_{j0} \exp(-\alpha_{j1}\tau), \tag{5.6}$$

$$\theta_j(\tau) = \theta_{j0}\theta_{j1}^2 \tau \exp(-\theta_{j1}\tau), \tag{5.7}$$

for positive scalars $\{(\alpha_{j0}, \alpha_{j1}, \theta_{j0}, \theta_{j1})\}_{j=H,F}$. The function $\alpha_j(\tau)$ that describes how the demand slope depends on bond maturity τ is a declining exponential. The function $\theta_j(\tau)$ that describes how the effect of demand shocks on the demand intercept depends on τ is the product of a declining exponential times τ. We take the demand parameters $(\alpha_{j0}, \alpha_{j1}, \theta_{j0}, \theta_{j1})$ to be the same across home and foreign, and drop the subscript j. In the case of θ_{j0} this is a normalization as we explain below. We set the maximum maturity T to infinity. The exponential specification of demand parallels that in Vayanos and Vila

(2021), and together with the assumption $T = \infty$ allow us to simplify the numerical solution method by using Laplace transforms, as shown in Appendix C.1.

Our estimation does not identify the two demand intercept parameters (θ_e, θ_0) and the arbitrageur risk-aversion coefficient a. The parameters (θ_e, θ_0) are not identified because they affect exchange rates and bond yields only through their products $(\theta_e\gamma_t, \theta_0\beta_{Ht}, \theta_0\beta_{Ft})$ with the demand factors. Hence, they affect the log-likelihood only through their products $(\theta_e\kappa_{\gamma,iH}, \theta_e\kappa_{\gamma,iF}, \theta_e\sigma_\gamma, \theta_0\sigma_\beta)$ with the parameters $(\kappa_{\gamma,iH}, \kappa_{\gamma,iF}, \sigma_\gamma, \sigma_\beta)$ that describe how short rates and Brownian shocks affect the demand factors. We focus on these products and drop $(\theta_e, \theta_0, \kappa_{\gamma,iH}, \kappa_{\gamma,iF}, \sigma_\gamma, \sigma_\beta)$ as separate parameters. The parameter a is not identified because it affects the log-likelihood only through its products $(a\theta_e\kappa_{\gamma,iH}, a\theta_e\kappa_{\gamma,iF}, a\theta_e\sigma_\gamma, a\theta_0\sigma_\beta, a\alpha_e, a\alpha_0)$ with the demand parameters. Intuitively, exchange rates and bond yields can be volatile if demand shocks are modest and arbitrageurs highly risk-averse, or if shocks are large and arbitrageurs modestly risk-averse. We bring additional information to calibrate a.

We use pre-set values for the parameters (α_1, θ_1) rather than estimating them. These parameters are not well identified by MLE because they have a weak effect on exchange rates and bond prices, and thus on the log-likelihood. They have a stronger effect on the distribution of trading volume across maturities, and can thus be identified using volume data. Volume is not affine in the state variables, however, and cannot be accommodated by our MLE estimation method. In our GMM estimation we use volume data and obtain estimates for (α_1, θ_1). The values for (α_1, θ_1) that we use in MLE are $(0.15, 0.3)$ in line with the GMM estimates and broadly in line with Vayanos and Vila (2021). Our results remain essentially unchanged for a wide range of other choices for (α_1, θ_1), as shown in Appendix C.2.

We use the US and the EZ as the home and foreign country, respectively, because they are roughly comparable in size and because a long time-series of zero-coupon bond yields for a large set of maturities is available. We use the German yield curve as the foreign yield curve. We use the Deutsche Mark as the foreign currency prior to introduction of the Euro in 01/1999. We use quarterly data on exchange rates and bond yields, thus setting $\Delta t = \frac{1}{4}$ (as in previous sections, the units of time t and maturity τ are years). Our sample starts in 06/1986, which is when zero-coupon bond yields are consistently available for maturities up to 20 years, and ends in 04/2021. We also estimate the model on different subsamples and find similar results as for our main sample, as shown in Appendix C.8. We source US yields from the Federal Reserve and German yields from the Bundesbank.[6]

We consider two cases for our $K \times 1$ vector p_t of observables. In our main case, $K = 5$ and p_t consists of US and German one- and ten-year yields and the Dollar-Euro log ex-

[6]https://www.federalreserve.gov/data/nominal-yield-curve.htm (Gurkaynak, Sack, and Wright (2007)) and https://www.bundesbank.de/en/statistics/money-and-capital-markets/interest-rates-and-yields/term-structure-of-interest-rates.

change rate. As we confirm with a sensitivity analysis in Appendix C.3, the data allow us to recover the model parameters as follows: the covariance matrix of innovations to the home and foreign one-year yields determines $(\sigma_{iH}, \sigma_{iF}, \sigma_{iH,iF})$; the covariance between innovations to the home one- and ten-year yields and its foreign counterpart determine $(\kappa_{iH}, \kappa_{iF})$; the covariances between innovations to home yields and the exchange rate, and their foreign counterparts, determine $(\kappa_\gamma, \theta_e\kappa_{\gamma,iF}, \theta_e\sigma_\gamma)$; the covariance matrix of innovations to the home and foreign ten-year yields determines $(\kappa_\beta, \theta_0\sigma_\beta, \alpha_0)$; and the variance of innovations to the exchange rate, together with the covariances between innovations to yields and the exchange rate, determine $(\theta_e\sigma_\gamma, \alpha_e)$. We additionally consider the case where $K = 21$ and p_t includes US and German yields with maturities from one to ten years and the Dollar-Euro log exchange rate. The results are similar to those in our main case, as shown in Appendix C.2.

We finally calibrate a. Since a is the coefficient of arbitrageur absolute risk aversion, it is equal to the coefficient γ of their relative risk aversion divided by their wealth W. We set $\gamma = 2$, in line with common estimates. An estimate for W can be derived by identifying arbitrageurs with hedge funds. The assets of hedge funds in the fixed-income, macro and balanced categories in 2020 were about 5% of US GDP in that year.[7] Taking US GDP as the numeraire, we can thus set $W = 5\%$. We use that value as a lower bound for W since arbitrageurs can include additional agents such as global banks and multinational corporations, and use 20% as an upper bound. The implied bounds for a are $2/5\% = 40$ and $2/20\% = 10$.

5.2 Model Fit

Table 1 reports the parameter estimates and standard errors. Innovations to the home short rate have somewhat higher standard deviation ($\sigma_{iH} = 1.16$) than to the foreign short rate ($\sqrt{\sigma_{iH,iF}^2 + \sigma_{iF}^2} = 0.936$), and are positively correlated (correlation $\frac{\sigma_{iH,iF}}{\sqrt{\sigma_{iH,iF}^2+\sigma_{iF}^2}} = 0.361$). The demand for foreign currency by currency traders responds negatively to the home short rate ($a\theta_e\kappa_{\gamma,iH} = -148$) and positively to the foreign short rate ($a\theta_e\kappa_{\gamma,iF} = 184$). Thus, a drop in the home short rate or a rise in the foreign short rate causes the demand for foreign currency to rise, holding the exchange rate constant. Innovations to the home and foreign short rates have similar persistence as innovations to the demand for foreign currency ($\kappa_{iH} = 0.145$, $\kappa_{iF} = 0.14$, and $\kappa_\gamma = 0.155$) and are all less persistent than innovations to bond demand ($\kappa_\beta = 0.068$).

The slope α_e of currency demand can be derived by dividing $a\alpha_e$ by a. It ranges from 7.689 ($=\frac{76.89}{10}$) for the lower bound of a to 1.922 ($=\frac{76.89}{40}$) for the upper bound. Thus, a 1% drop in the exchange rate raises the demand for foreign currency by an amount ranging

[7] https://www.barclayhedge.com/solutions/assets-under-management/hedge-fund-assets-under-management/

Parameter	Value	Std. Err.
σ_{iH}	1.163	0.076
σ_{iF}	0.873	0.058
$\sigma_{iH,iF}$	0.338	0.081
κ_{iH}	0.145	0.058
κ_{iF}	0.140	0.046
κ_{β}	0.068	0.055
κ_{γ}	0.155	0.100
$a\theta_e\kappa_{\gamma,iH}$	-148.1	116.9
$a\theta_e\kappa_{\gamma,iF}$	184.2	130.6
$a\theta_0\sigma_{\beta}$	884.8	173.9
$a\theta_e\sigma_{\gamma}$	941.6	464.8
$a\alpha_0$	5.379	3.096
$a\alpha_e$	76.89	37.52

Table 1: Estimated model parameters

from 7.7% (= 7.7 × 1%) to 1.9% of US GDP. The slope parameter α_0 of bond demand can likewise be derived by dividing $a\alpha_0$ by a, and ranges from 0.538 to 0.134. To interpret this coefficient, consider a uniform 0.1% rise in home or foreign bond yields. This translates to a price drop of the corresponding τ-year bond by $\tau \times 0.1\%$, which for a ten-year bond is 1%, the same as for the exchange-rate exercise. The aggregate bond demand across maturities rises by an amount ranging from 2.391% ($= \int_0^\infty \alpha_0 \exp(-\alpha_1\tau)\tau d\tau \times 0.1\% = \frac{\alpha_0}{\alpha_1^2} \times 0.1\% = \frac{0.538}{0.15^2} \times 0.1\%$) of US GDP to 0.598%. By comparison, Krishnamurthy and Vissing-Jorgensen (2012) estimate that a 0.1% drop in the spread between AAA-rated US corporate bonds and US government bonds raises government bond demand by 5.9% of US GDP. Their estimate is about four times as large as the midpoint of ours. This discrepancy may arise because an increase in government bond yields in our estimation can be accompanied by an increase in corporate bond yields (which mitigates the increase in the spread). Another estimate comes from Koijen and Yogo (2020), who find that a 1% price drop in the price of long-maturity bonds (maturity of one year or longer) raises their demand by foreign investors by 1.9%. This estimate is not directly comparable to ours because we do not distinguish whether investors for a country's bonds are home or foreign.

Our MLE estimation strategy delivers precise estimates for most parameters. The least precisely estimated parameters are $(\kappa_\beta, \kappa_\gamma, \theta_e\kappa_{\gamma,iH}, \theta_e\kappa_{\gamma,iF})$, which determine the persistence of demand shocks and the effect of short-rate shocks to currency demand. The precision of the parameter estimates matters primarily for our results on return predictability in Section 5.3 and monetary policy transmission in Section 5.4. We compute confidence intervals for these results using the covariance matrix of parameter estimates.

5.3 Currency and Bond Returns

Variance Decomposition of Returns: Figure 1 shows variance decompositions of currency and bond returns within our estimated model. The top row concerns home bond returns, the middle row foreign bond returns and the bottom row currency returns (all in logs). In each row, the left column concerns returns over a one-quarter horizon and the right column returns over a five-year horizon. In the top and middle rows, each color represents the fraction of the variance of bond returns explained by each factor, with the order from bottom to top being $(i_{Ht}, i_{Ft}, \beta_{Ht}, \beta_{Ft}, \gamma_t)$. In the bottom rows, each bar represents the fraction of the variance of currency returns explained by each factor, with the order from left to right being $(i_{Ht}, i_{Ft}, \beta_{Ht}, \beta_{Ft}, \gamma_t)$. All variance decompositions concern the surprise component of returns, derived by subtracting conditional expected returns from realized returns. We examine the variation of conditional expected returns later in this section. The variance decompositions of bond returns are plotted as function of the maturity at the end of the return horizon. For example, the decomposition of the six-year bond's five-year return is shown at the one-year x-axis coordinate. The calculations of currency and bond returns and of the model-implied variance decompositions are in Appendix C.5.

Figure 1 implies that the demand factors drive most of the variation of the returns on currency and long-maturity bonds. In the case of currency returns, the currency demand factor drives approximately 90% of the variation over the one-quarter horizon and 70% over the five-year horizon. The home and foreign short rates drive most of the remaining variation. In the case of bond returns, over both return horizons, the home and foreign bond demand factors drive approximately 80% of the variation for bonds with ten-year maturity (where maturity is measured at the end of the horizon) and well over 90% of the variation for bonds with maturities of twenty years and longer. Most of the variation of a country's long-maturity bond returns is driven by the local bond demand factor. Yet, the bond demand factor in the other country plays a role as well, indicating significant cross-country spillovers. The home and foreign short rates drive most of the remaining variation.

Figure 1 implies additionally a disconnect between the exchange rate and yields on long-maturity bonds. The exchange rate moves primarily because of shocks to the currency demand factor, with shocks to the bond demand factors having a small effect. Conversely, yields of long-maturity bonds move primarily because of shocks to the bond demand factors, with shocks to the currency demand factor having a small effect. Long-maturity bond yields, however, are connected across countries: yields in one country are influenced significantly by shocks to the bond demand factor in the other country.

Why is the exchange rate disconnected from yields on long-maturity bonds? And why are long-maturity bond yields connected across countries despite their disconnect from

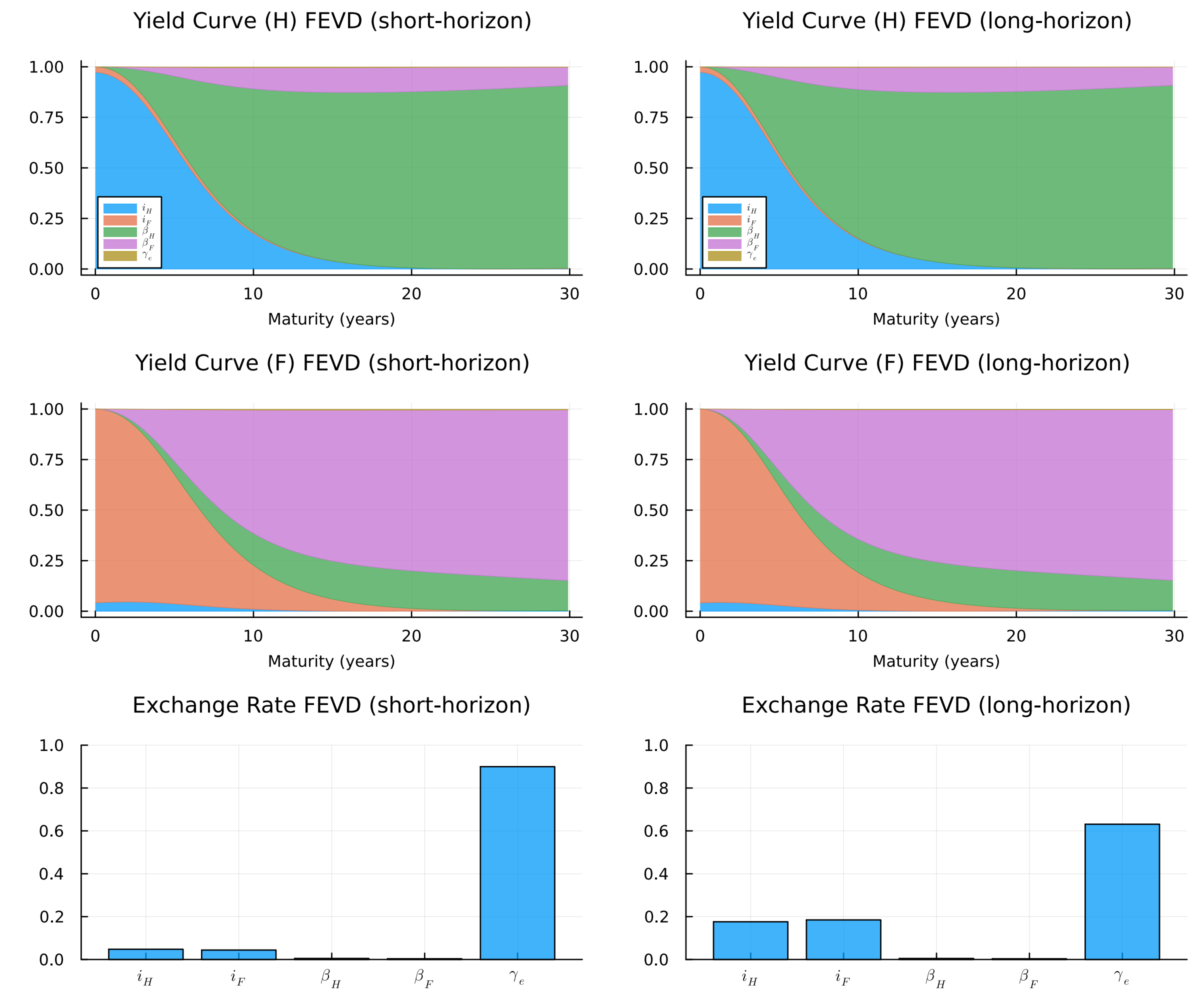

Figure 1: Variance decomposition of currency and bond returns

the exchange rate? Key to the answers is the comovement between the exchange rate and bond yields that the demand factors generate. Recall from Section 4 that in the absence of demand risk, an unanticipated one-off drop in investor demand for home bonds raises home bond yields, raises foreign bond yields when the demand of currency traders is price-elastic ($a_e > 0$), and causes the foreign currency to depreciate (Proposition 4.7). Moreover, the transmission of the home bond demand shock to foreign bond yields occurs through the currency market: arbitrageurs hold more home bonds in response to the drop in investor demand, hold less foreign currency because of their increased exposure (through home bonds) to a rise in the home short rate, and hold fewer foreign bonds because of their decreased exposure (through foreign currency) to a drop in the foreign short rate. Thus, if shocks to the home bond demand factor have a weak effect on the exchange rate, they should also have a weak effect on foreign bond yields.

Introducing the bond demand factors strengthens the transmission of a home bond demand shock to foreign bond yields but does not affect the shock's transmission to the exchange rate. Indeed, because bond demand shocks move home and foreign bond yields in the same direction, they induce positive comovement between home and foreign bond returns, additional to the positive comovement induced by short-rate shocks in Section 4. Because of this additional comovement, a drop in investor demand for home bonds has a larger negative effect on foreign bond returns than in Section 4. No additional such comovement is induced between bond and currency returns because home and foreign bond demand shocks move currency returns in opposite directions. We demonstrate these mechanisms in Section 5.4, where we examine how the effects of QE depend on the standard deviation σ_β of innovations to the bond demand factors (Figure 6).

A positive correlation between home and foreign bond returns arises more mechanically when home and foreign short rates are positively correlated, as is the case in our estimated model. Even when short rates are independent, however, long-maturity bond yields remain positively correlated across countries. Long-maturity bond yields become independent across countries only when short rates are independent and $\alpha_e = 0$.

With the bond demand factors but without the currency demand factor, the exchange rate is connected weakly to long-maturity bond yields and strongly to the short rates. Introducing the currency demand factor weakens the latter connection as well because currency demand shocks generate variation in the exchange rate that is unrelated to the short rates. Additionally, because currency demand shocks generate opposite movements in home and foreign bond yields, as shown in Section 4, they weaken the strong positive comovement between long-maturity bond yields across countries. They do not undo that comovement, however. We demonstrate these mechanisms in Section 5.4, where we examine how the effects of QE depend on the standard deviation σ_γ of innovations to the currency demand factor (Figure 6).

Our result that long-maturity bond yields are connected across countries but are disconnected from the exchange rate holds in the data. In Appendix C.6 we show that the empirical correlations between quarterly changes to the exchange rate, home bond yields and foreign bond yields are close to the model-implied ones. It is because of the low empirical correlation between changes to the exchange rate and bond yields that our estimation yields near-independent currency and bond demand factors even when we allow for correlation in Appendix C.2. The demand factors can be interpreted as shocks affecting different types of agents or intermediaries.

The disconnect between the exchange rate and long-maturity bond yields in our estimated model is consistent with the findings in Chernov and Creal (2023) and Chernov, Haddad, and Itskokhi (2024) that bond returns span only a small fraction of the variation in currency returns. Kekre and Lenel (2024) find that spanning improves for long-horizon returns, and develop a macroeconomic model with discount-rate shocks that delivers more spanning. Spanning in our estimated model is comparable to the empirical findings in these papers for short-horizon returns. Spanning in our model for long-horizon returns is larger than for short-horizon returns, as can be seen by comparing the bottom left and right panels in Figure 1, but is not as large as in Kekre and Lenel (2024).

Return Predictability Regressions: We next examine the implications of our estimated model for the predictability of currency and bond returns. We do so by running common regressions in the asset pricing literature and comparing the empirical coefficients computed within our sample to the coefficients implied by our model. The empirical coefficients are the red circles in Figures 2 and 3, and the model-implied coefficients are the blue solid lines. The black dashed lines show the UIP benchmark in Figure 2 and the EH benchmark in Figure 3. Our model generates these benchmarks when setting the arbitrageur risk-aversion coefficient a to zero (Corollary B.1). Vertical lines and shaded areas are 90% confidence intervals for the empirical and model-implied coefficients, respectively. The calculations of the model-implied coefficients are in Appendix C.7.

Figure 2 reports coefficients for various types of UIP regressions. The top left panel concerns the hybrid UIP regression of Lustig, Stathopoulos, and Verdelhan (2019, LSV), in which the return over horizon $\Delta\tau$ of the hybrid CCT constructed using bonds with maturity τ is regressed on the foreign-minus-home $\Delta\tau$-year yield differential. This regression nests as a special case, for $\tau = \Delta\tau$, the standard UIP regression of Bilson (1981) and Fama (1984). Under UIP, the LSV coefficient should be zero. The empirical coefficients are positive for short maturities, consistent with Bilson (1981) and Fama (1984). They decline with maturity and become statistically insignificant for long maturities. This is consistent with LSV, although LSV's coefficients, computed over multiple currency pairs rather than over only Dollar/Euro as in our estimation, are closer to zero. The model-implied coefficients are close to their empirical counterparts because of the mechanisms in Propositions 4.1 and 4.5.

The top right panel in Figure 2 concerns the long-horizon UIP regression of Chinn and Meredith (2004, CM), in which the rate of foreign currency depreciation over horizon $\Delta\tau$ is regressed on the foreign-minus-home $\Delta\tau$-year yield differential. Under the UIP, the CM coefficient should be one. The empirical coefficients are negative for short maturities. They increase with maturity and reach one for long maturities, consistent with CM. The model-implied coefficients increase towards one but do not reach it closely. This reflects the lack of convergence of the long-horizon CCT's expected return to zero because not all eigenvalues of M have positive real parts in our estimated model. Nevertheless, the model-implied CM coefficients are within the confidence intervals of the empirical coefficients for the ten-year maturity (and for maturities one through six).

The bottom two panels concern regressions run in Chen and Tsang (2013), Lloyd and Marin (2020) and Chernov and Creal (2021), in which the rate of foreign currency depreciation over horizon $\Delta\tau$ is regressed on the foreign-minus-home $\Delta\tau$-year yield differential (level—same regressor as in CM), and the foreign-minus-home differential of the slope of the term structure (slope). Under UIP, the level coefficient should be one and the slope coefficient should be zero. As with the CM regression, the coefficients using only one

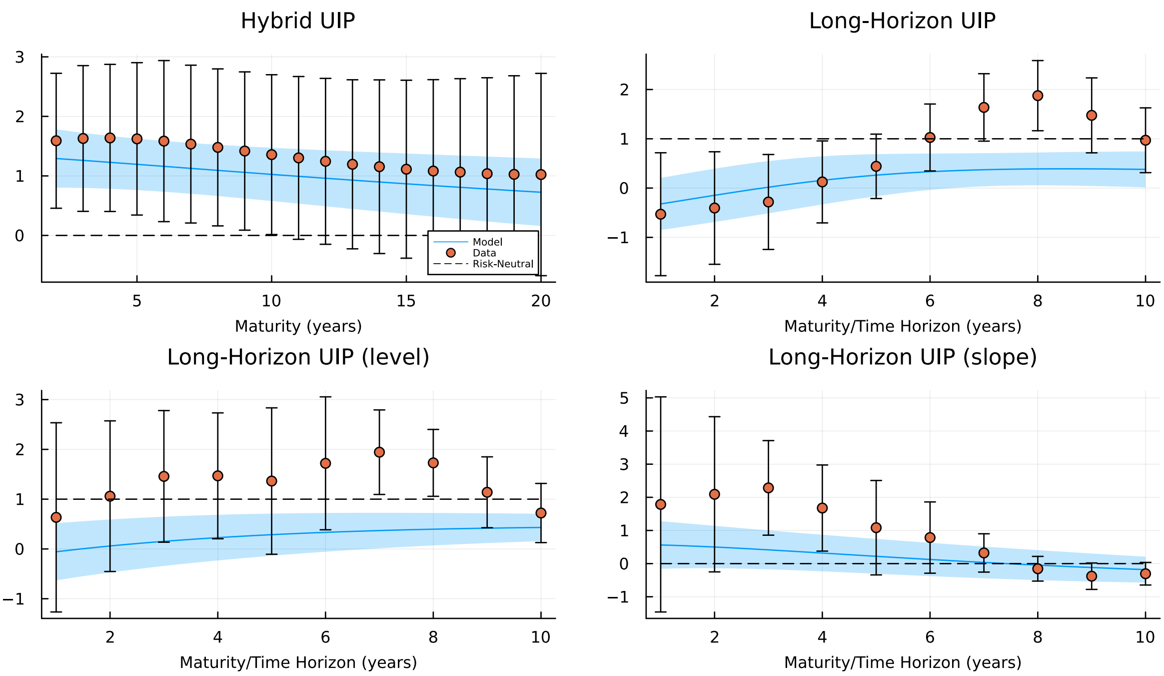

Figure 2: Coefficients of UIP regressions

currency pair are imprecisely estimated, but the point estimates are consistent with the literature. As in the literature, the slope coefficient is positive, meaning that for a given yield differential, the foreign-minus-home CCT is less profitable when the foreign-minus-home slope differential is larger. Our model implies a positive slope coefficient, as in the data. Indeed, suppose that investor demand for foreign bonds is temporarily low. This pushes up foreign bond yields, raising the foreign-minus-home slope differential. It also causes the foreign currency to appreciate temporarily (Proposition 4.7), and its future expected return to decline. As in the data, the predictability of slope in our model is primarily over short and medium maturities. Overall, our model matches well the UIP regression evidence.

Figure 3 reports coefficients for various types of EH regressions. The top left and top right panels concern the Fama and Bliss (1987, FB) regression in the home and foreign country, respectively. FB regress the excess return over horizon $\Delta\tau$ of the bond with maturity τ on the slope of the term structure, measured by the difference between the forward rate between maturities $\tau - \Delta\tau$ and τ, and the $\Delta\tau$-year spot rate (yield). The bottom left and bottom right panels concern the Campbell and Shiller (1991, CS) regression in the home and foreign country, respectively. CS regress the change over horizon $\Delta\tau$ in the yield of a bond with initial maturity τ on a scaled difference between the spot rates for maturities τ and $\Delta\tau$. Under the EH, the FB coefficient should be zero and the CS coefficient should be one. The empirical coefficients are consistent with the findings of FB and CS: the FB coefficient is positive and increasing with maturity, and the CS coefficient is negative and decreasing with maturity.

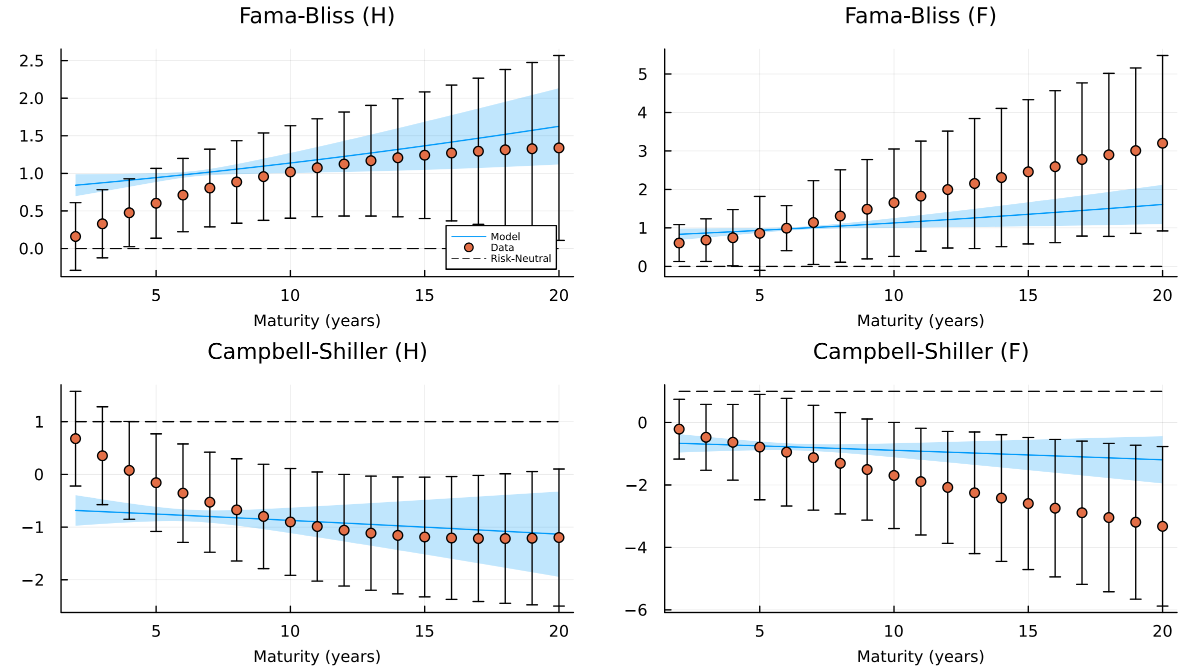

Figure 3: Coefficients of EH regressions

The model-implied FB and CS coefficients have the same sign as their empirical counterparts. In the absence of demand risk, the sign is the same because of the mechanisms in Proposition 4.2. Demand risk preserves the sign because when bond demand is low, the slope of the term structure is high and so is the BCT's expected return. The model-implied FB coefficients for both countries are slightly below one for short maturities and rise above one for longer maturities. Conversely, the model-implied CS coefficients are between zero and minus one for short maturities and drop to below minus one for longer maturities. The model-implied FB and CS coefficients in the US match closely their empirical counterparts, except for short maturities, where they are overstate EH violations. The model-implied FB and CS coefficients in Germany match closely their empirical counterparts for short maturities but understate EH violations for longer maturities. In both cases, the model-implied coefficients have the same monotonicity as in the data but the extent of monotonicity is weaker than in the data.

5.4 Monetary Policy Transmission

Conventional Policy: We next use our estimated model to study the domestic and international transmission of monetary policy. We start with conventional policy, and consider a cut to the short rate by the central bank in country j. We assume that the cut is unanticipated, occurs at time zero, is unwound over time at a rate κ_{ij}^{MP} possibly different from κ_{ij} and affects the currency demand factor according to $\kappa_{\gamma,ij}$. We model the short-rate and currency demand factor movements induced by the cut as separate, additive components $\Delta i_{jt} = \Delta i_{j0} e^{-\kappa_{ij}^{MP} t}$ and $\Delta \gamma_t = \Delta i_{j0} \frac{\kappa_{\gamma,ij}}{\kappa_\gamma - \kappa_{ij}^{MP}} \left(e^{-\kappa_\gamma t} - e^{-\kappa_{ij}^{MP} t} \right)$ of the corresponding processes. We set $\Delta i_{j0} = -0.25$, implying a cut of 25 basis points (bps), and $\kappa_{ij}^{MP} = 0.75$, implying a half-life of the cut of about a year.

The top left and top right panels of Figure 4 show, respectively, how a cut to the home short rate affects the home and foreign term structures at time zero and how it affects the exchange rate over time. The bottom left and right panels show the same for a cut to the foreign short rate. The home term structure is shown in blue and the foreign term structure in red. Exchange-rate movements are measured as percentage price changes. The figure includes 90% confidence intervals.

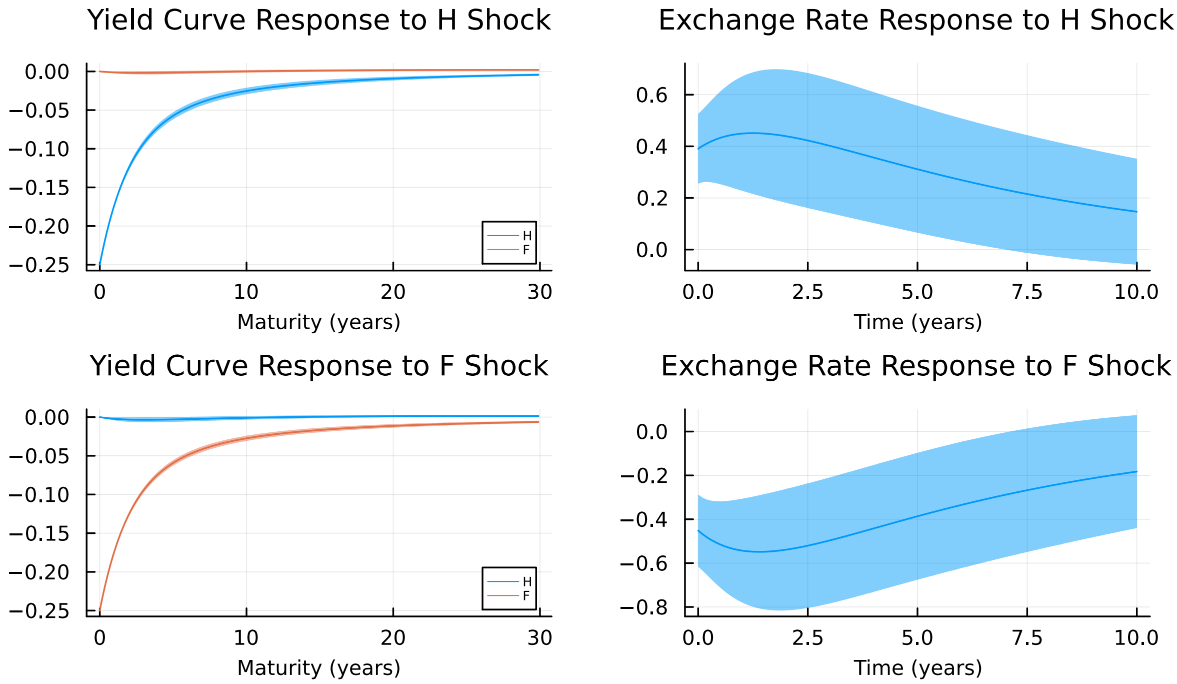

Figure 4: Conventional monetary policy – Short rate cut

The rate cut affects the domestic term structure but has essentially no effect on the foreign term structure. This result is consistent with Georgiadis and Jarocinski (2023), who find that the effects of US conventional policy on foreign term premia are negligible. The international transmission of conventional policy is weak because of the disconnect between the exchange rate and long-maturity bond yields. Recall from Section 4 that in the absence of demand risk, a cut to the home short rate is transmitted to foreign bond yields through the currency market: arbitrageurs hold more foreign currency because the CCT becomes more profitable, and hold more foreign bonds because of their increased exposure (through currency) to a drop in the foreign short rate. In the presence of demand risk and under our estimated parameters, short rates drive a small fraction of currency returns. Therefore, arbitrageurs make limited use of the foreign bond market to hedge their currency position, and the effect of a cut to the home short rate on foreign bond yields is small.

In the presence of positively correlated short rates or of demand risk, there is an additional transmission channel relative to Section 4. In response to a cut to the home short rate, arbitrageurs hold more home bonds because the home BCT becomes more profitable. Since home and foreign bonds are positively correlated when short rates are positively correlated or when there is demand risk (and $a_e > 0$), arbitrageurs become less willing to hold foreign bonds. This effect works in the opposite direction to that in Section 4, further weakening the transmission.

The response of the exchange rate to the rate cut is hump-shaped. Following a cut to the home short rate, the exchange rate jumps up (i.e., the foreign currency appreciates) by 0.4%, then rises further by 0.075% over the next two years, and then declines gradually to its pre-cut value. The exchange-rate response is hump-shaped because demand for foreign currency rises gradually following the cut to the home short rate ($\kappa_{\gamma,iH} < 0$). The demand effect is dominant: in its absence, the exchange rate would jump up by roughly 0.15%, only one-half of the effect under UIP ($\frac{0.25\%}{\kappa_{ij}^{MP}} = 0.33\%$).

Quantitative Easing: We next turn to unconventional policy, and consider QE purchases by the central bank in country j. We assume that the purchases are unanticipated, occur at time zero, and are unwound over time at a rate $\kappa_{\beta j}^{QE}$ possibly different from $\kappa_{\beta j}$. We model the addition to the central bank's position in the bond with maturity τ as a separate, additive component $\theta_j^{QE}(\tau)\Delta\beta_{jt}$ of the demand-intercept process, where $\theta_j^{QE}(\tau)$ has the exponential form (5.7) and $\Delta\beta_{jt} = \Delta\beta_{j0}e^{-\kappa_{\beta j}^{QE}t}$. We allow the parameters $(\theta_0^{QE}, \theta_1^{QE})$ to differ from (θ_0, θ_1), and normalize $\Delta\beta_{j0}$ to one. We set $\theta_1^{QE} = 0.2$, implying that QE purchases are maximized at the five-year maturity ($\frac{1}{\theta_1^{QE}} = 5$), $\theta_0^{QE} = 0.1$, implying that purchases are 10% of GDP ($\int_0^\infty \theta_0\theta_1^2 \exp(-\theta_1\tau)\tau d\tau = \frac{\theta_0\theta_1^2}{\theta_1^2} = \theta_0 = 0.1$), and $\kappa_{\beta j}^{QE} = 0.1$, implying a half-life of purchases of about seven years.

The top left and top right panels of Figure 5 show, respectively, how QE purchases in the home country affect the home and foreign term structures at time zero and how they affect the exchange rate over time. The bottom left and right panels show the same for QE purchases in the foreign country. The coloring, units and confidence intervals are as in Figure 4. Figure 5 sets arbitrageur risk aversion a to 40. When $a = 10$, the effects are one-quarter of those in the figure.

QE purchases have sizeable effects on domestic yields: they reduce the ten-year yield by 35-45 bps and the thirty-year yield by 65-80 bps. These estimates, derived for $a = 40$, align with the literature: according to Wiliams (2014), a consensus estimate is that QE purchases of 10% of GDP reduce the ten-year yield by 35-65 bps.

QE purchases have sizeable effects on foreign yields as well: they move by one-half to one-third as much as their domestic counterparts. Hence, conventional and unconventional policy differ sharply in their transmission to foreign bond yields: spillovers are small for the former and large for the latter. This result is consistent with Georgiadis and Jarocinski (2023), who find that the effects of US QE/QT on foreign term premia are large and exceed

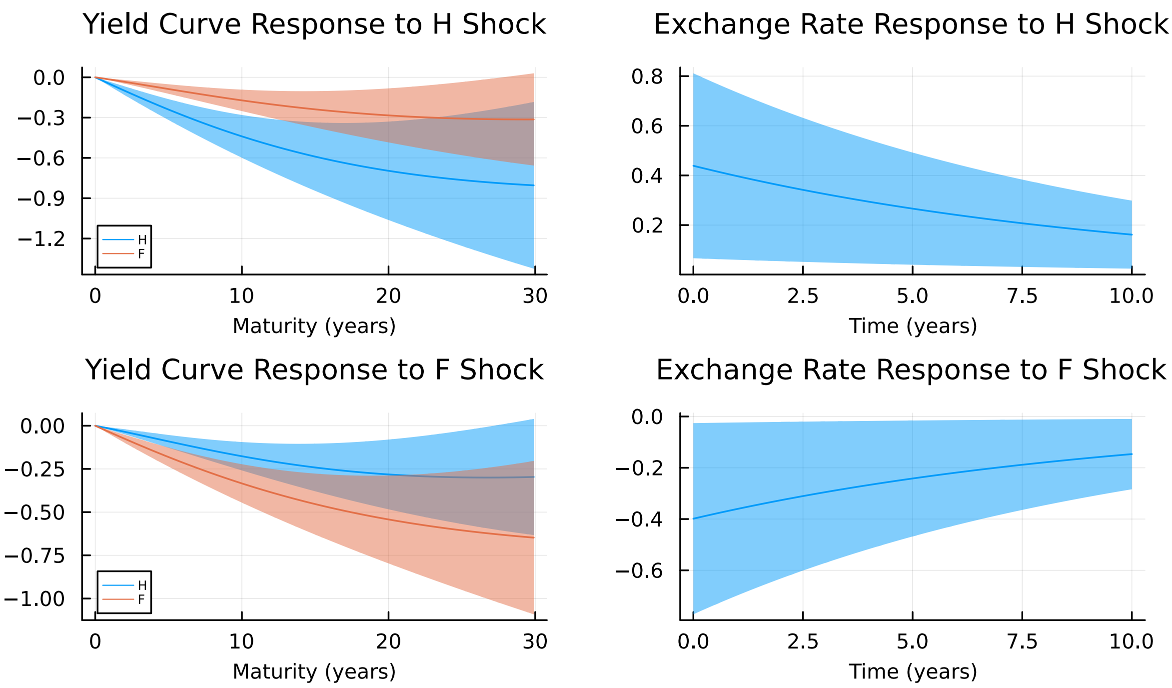

Figure 5: Unconventional monetary policy – Bond purchases

those of conventional policy (which are negligible) and of forward guidance. To explain the intuition, recall from Section 5.3 that foreign bond yields move because of the foreign bond demand factor and short rate and because of the home bond demand factor (Figure 1). Since monetary policy does not affect directly foreign bond demand and the short rate, differences in its transmission to foreign bond yields should be traced to differences in how it affects domestic bond demand. Unconventional monetary policy has a large effect on domestic bond demand because it targets it directly: Figure 5 concerns QE purchases of 10% of GDP. Conventional policy has a smaller effect on domestic demand because it targets it only indirectly: following the rate cut of 25 bps in Figure 4, investor demand for domestic bonds drops by only 0.2% of GDP. We illustrate the effects of conventional and unconventional policy on agents' holdings of currency and bonds in Appendix C.8 (Figures C.2 and C.3). Appendix C.8 also includes counterparts of Figures 4 and 5 constructed using our estimates from different subsamples of the data (Figures C.4 and C.5).

The effect of unconventional policy on the exchange rate is comparable to that of conventional policy, consistent with Georgiadis and Jarocinski (2023). QE in the home country causes the foreign currency to appreciate by 0.45%, while QE in the foreign country causes the foreign currency to depreciate by 0.4%. These effects are smaller than those on bond returns (a 35-45 bps drop in the ten-year yield translates to a 3.5-4.5% price rise) because of the disconnect between the exchange rate and bond yields. Note that since the effect of unconventional policy on the exchange rate is small relative to that on the long-maturity cross-country yield differential (45 bps for the thirty-year yield), demand shocks impact the long-horizon CCT's expected return, even for long maturities.

Figure 6 shows comparative statics of the effects of QE. We vary the standard deviation σ_β of innovations to the bond demand factors and σ_γ of innovations to the currency demand factor. Each parameter varies from zero to its estimated value, which corresponds to one in the x-axis by normalizing the units. All parameters except for the one that varies are set to their estimated values. The figure shows the ten-year yield and the exchange rate. Results in the top row are for σ_β and in the bottom row for σ_γ. Additional comparative statics, with respect to $(\alpha_0, \alpha_e, \sigma_{iH,iF})$, are in Appendix C.8 (Figure C.6).

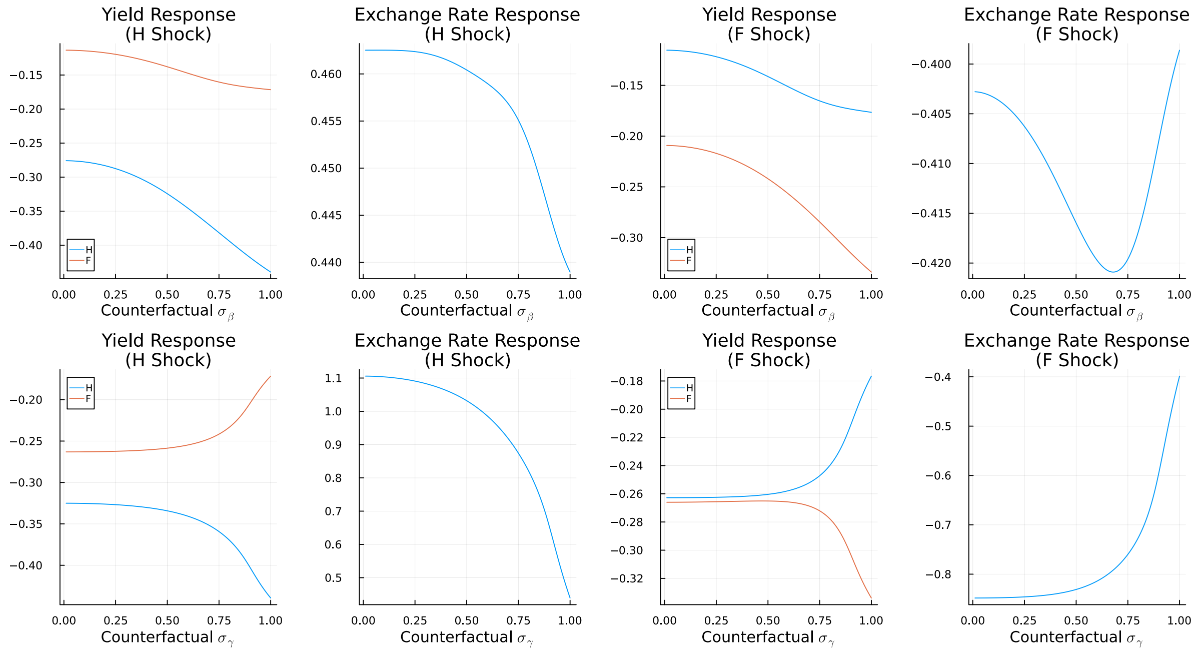

Figure 6: Comparative statics of effects of QE

When σ_β increases, QE has stronger effects on domestic bond yields. This is because bond returns become more volatile, and thus arbitrageurs require a larger price increase to sell (or short-sell) domestic bonds to the central bank. QE has stronger effects on foreign bond yields as well. This is because the covariance between bond returns across countries increases, and thus arbitrageurs find foreign bonds more valuable to replace the domestic bonds that they sell to the central bank. The effects of QE on the exchange rate remain roughly unchanged because the covariance between currency and bond returns does not change.

When σ_γ increases, QE has stronger effects on domestic bond yields and weaker effects on foreign bond yields and the exchange rate. The effects on the exchange rate are weaker because short rates drive a smaller fraction of currency returns and thus arbitrageurs make more limited use of the currency market to hedge a change in their position in domestic bonds. The effects on foreign bonds are weaker because currency demand shocks generate opposite movements in home and foreign bond yields, thus weakening the positive covariance between bond yields across countries.

Foreign Exchange Interventions: We finally examine the effects of foreign exchange (FX) interventions. We assume that the home or foreign central bank purchases foreign currency and that these purchases are unanticipated, occur at time zero, and are unwound over time at a rate κ_γ^{FX} possibly different from κ_γ. We model the net addition to the central bank's position in foreign currency as a separate, additive component $\theta_e^{FX}\Delta\gamma_t$ of the currency demand factor process, where $\Delta\gamma_t = \Delta\gamma_0 e^{-\kappa_\gamma^{FX} t}$. We normalize $\Delta\gamma_0$ to one, set $\kappa_\gamma^{FX} = 0.75$, implying a half-life of the FX intervention of about one year as in the conventional monetary policy exercise, and $\theta_e^{FX} = 0.1$, implying that the FX intervention is 10% of GDP as in the unconventional monetary policy exercise.

The top left and top right panels of Figure 7 show, respectively, how the FX intervention affects the home and foreign term structures at time zero and how they affect the exchange rate over time. The coloring, units and confidence intervals are as in Figure 4. Figure 5 sets arbitrageur risk aversion a to 40. When $a = 10$, the effects are one-quarter of those in the figure.

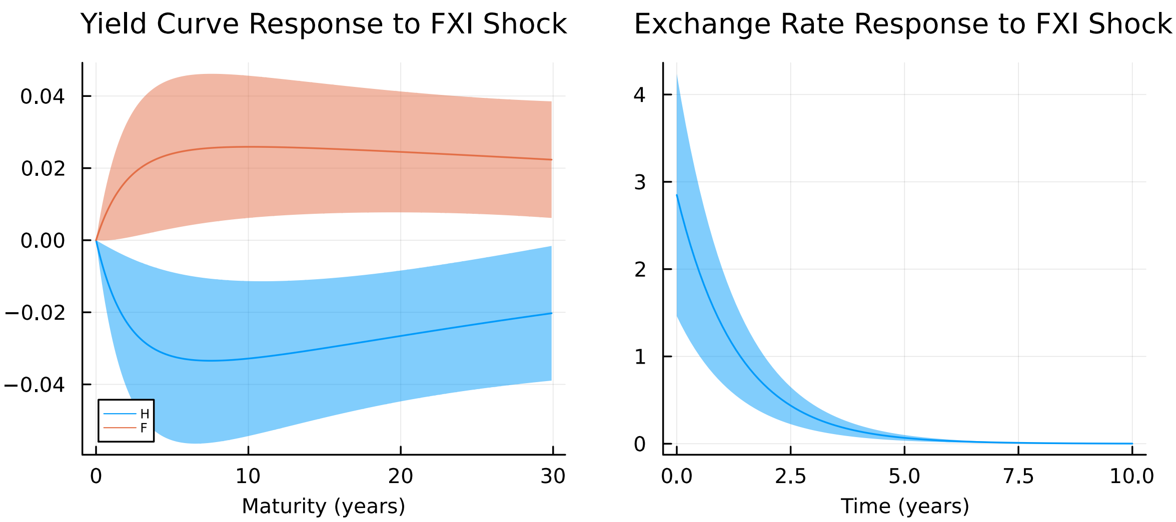

Figure 7: Foreign currency intervention

The FX intervention has a large effect on the exchange rate: the foreign currency appreciates by nearly 3% on impact, and depreciates gradually to its pre-intervention value as the intervention is unwound. On the other hand, the intervention's effects on the home and foreign yield curve are weak: ten-year yields change by 2-3 bps and thirty-year yields by 2 bps. The effects on yields are weak because the exchange rate and bond yields are disconnected.

6 Concluding Remarks

We develop a two-country model in which currency and bond markets are populated by different investor clienteles, and segmentation is partly overcome by global arbitrageurs with limited capital. We show that the combination of limited arbitrage and price-elastic clienteles gives rise to the empirically documented violations of UIP and EH, and to the ways in which the violations are connected. The resulting risk premia in currency and bond markets, which are time-varying and connected, play a key role in the transmission of monetary policy. It is through changes in risk premia in both markets that QE purchases lower domestic and foreign bond yields and depreciate the currency, and that short-rate cuts lower foreign bond yields. Movements in the exchange rate in our model are almost uncorrelated with movements in long-maturity bond yields, and yet the currency market is instrumental in transmitting bond demand shocks across countries. In particular, the effects of QE/QT on foreign bond yields are sizeable, and stronger than those of conventional policy. At the same time, because of the disconnect between the exchange rate and bond yields, the transmission of QE/QT to the exchange rate is weaker, and foreign exchange interventions by central banks have strong effects on the exchange rate but weak effects on bond yields.

A natural next step is to embed our model within a fuller macroeconomic setting, in which short rates, inflation, and the demands of currency traders and bond investors are determined endogenously. Ray (2019) and Ray, Droste, and Gorodnichenko (2024) integrate elements of bond-market segmentation and limited arbitrage into closed-economy macroeconomic settings. Doing the same in multi-country settings entails additional challenges, such as connecting the flow demand for foreign assets arising from households' expenditure switching between domestic and foreign goods to the stock demand by currency traders, and connecting the capital gains and losses of arbitrageurs to countries' net foreign assets. We address these challenges in ongoing work (Gourinchas, Ray, and Vayanos (2024, GRV1)), where we develop a two-country New Keynesian model in which currency and bond markets are segmented for households and are partly integrated by arbitrageurs with limited capital.

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Online Appendix for

"A Preferred-Habitat Model of Term Premia, Exchange Rates, and Monetary Policy Spillovers"

A Forwards and Swaps

We show that the equilibrium with a currency forward market is equivalent to one without it but with different demands for foreign assets, home bonds and foreign bonds. The equivalence result extends to swaps because they are portfolios of forwards.

We model the demand for currency forwards as follows. Currency traders with preferences for currency forwards with maturities in $[\tau, \tau + d\tau]$ are in measure $d\tau$, and their demand, expressed in units of the home currency, is

$$Z_{et}^{(\tau)} = -(\zeta_e(\tau) + \theta_e(\tau)\gamma_t), \tag{A.1}$$

where $(\zeta_e(\tau), \theta_e(\tau))$ are functions of τ.

Since global arbitrageurs can trade costlessly foreign currency, home bonds, and foreign bonds, Covered Interest Parity (CIP) holds. Moreover, buying $Z_{et}^{(\tau)}$ of the currency forward with maturity τ is equivalent to the combination of (i) selling $Z_{et}^{(\tau)}$ of the home bond with maturity τ for home currency, (ii) selling $Z_{et}^{(\tau)}$ of home currency for foreign currency, and (iii) selling $Z_{et}^{(\tau)}$ of foreign currency for the foreign bond with maturity τ. Hence, the equilibrium with the currency forward market is equivalent to one without it but with $Z_{et}^{(\tau)}$ added to the demand (2.4) for foreign assets, $-Z_{et}^{(\tau)}$ added to the home bond demand (2.5), and $Z_{et}^{(\tau)}$ added to the foreign bond demand (2.5). The demand for foreign assets becomes

$$Z_{et} + \int_0^T Z_{et}^{(\tau)} d\tau = -\alpha_e \log(e_t) - (\zeta_{et} + \theta_e\gamma_t) - \int_0^T (\zeta_e(\tau) + \theta_e(\tau)\gamma_t) d\tau$$

instead of Z_{et}. The demand for country j bonds with maturity τ becomes

$$Z_{jt}^{(\tau)} + (-1)^{1_{\{j=H\}}} Z_{et}^{(\tau)} = -\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) - (\zeta_j(\tau) + \theta_j(\tau)\beta_{jt}) - (-1)^{1_{\{j=H\}}} (\zeta_e(\tau) + \theta_e(\tau)\gamma_t)$$

instead of $Z_{jt}^{(\tau)}$. Forwards induce a negative correlation between the demand for foreign assets and for home bonds, and a positive correlation between the demand for foreign assets and for foreign bonds.

B Proofs

Proposition B.1 characterizes the equilibrium in Section 3. We denote by $(\mathcal{E}_{iH}, \mathcal{E}_{iF}, \mathcal{E}_\gamma, \mathcal{E}_{\beta H}, \mathcal{E}_{\beta F})$ the five columns of the 5×5 identity matrix.

Proposition B.1. *When arbitrage is global, the exchange rate e_t is given by (3.1) and bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (3.2), with (A_e, C_e) solving*

$$MA_e - \mathcal{E}_{iH} + \mathcal{E}_{iF} = 0, \tag{B.1}$$

$$-A_e^\top \Gamma \bar{q} - (\pi_F - \pi_H) + \frac{1}{2} A_e^\top \Sigma\Sigma^\top A_e = A_e^\top \lambda_C, \tag{B.2}$$

and $(A_j(\tau), C_j(\tau))$ solving

$$A_j'(\tau) + MA_j(\tau) - \mathcal{E}_{ij} = 0, \tag{B.3}$$

$$C_j'(\tau) - A_j(\tau)^\top \Gamma \bar{q} + \frac{1}{2} A_j(\tau)^\top \Sigma\Sigma^\top \left(A_j(\tau) + 2A_e 1_{\{j=F\}}\right) = A_j(\tau)^\top \lambda_C, \tag{B.4}$$

with the initial conditions $A_j(0) = C_j(0) = 0$, and

$$\begin{aligned} M \equiv \Gamma^\top - a \Bigg(&(\theta_e \mathcal{E}_\gamma - \alpha_e A_e) A_e^\top \\ &+ \sum_{j=H,F} \int_0^T (\theta_j(\tau)\mathcal{E}_{\beta j} - \alpha_j(\tau) A_j(\tau)) A_j(\tau)^\top d\tau \Bigg) \Sigma\Sigma^\top, \end{aligned} \tag{B.5}$$

$$\lambda_C \equiv a\Sigma\Sigma^\top \left((\zeta_e - \alpha_e C_e) A_e + \sum_{j=H,F} \int_0^T (\zeta_j(\tau) - \alpha_j(\tau) C_j(\tau)) A_j(\tau) d\tau \right). \tag{B.6}$$

Proof of Proposition B.1: Using the definitions of $(\mathcal{E}_{iH}, \mathcal{E}_{iF}, \mathcal{E}_\gamma, \mathcal{E}_{\beta H}, \mathcal{E}_{\beta F})$, we can write (3.16) as

$$\begin{aligned} &a\Sigma\Sigma^\top \left(A_e (\theta_e \mathcal{E}_\gamma - \alpha_e A_e)^\top + \sum_{j=H,F} \int_0^T A_j(\tau) (\theta_j(\tau)\mathcal{E}_{\beta j} - \alpha_j(\tau) A_j(\tau))^\top d\tau \right) q_t \\ &+ a\Sigma\Sigma^\top \left((\zeta_e - \alpha_e C_e) A_e + \sum_{j=H,F} \int_0^T (\zeta_j(\tau) - \alpha_j(\tau) C_j(\tau)) A_j(\tau) \right) \\ &= -(M - \Gamma^\top)^\top q_t + \lambda_C, \end{aligned} \tag{B.7}$$

where the second step follows from the definitions of (M, λ_C). We next substitute $(\mu_{et}, \{\mu_{jt}^{(\tau)}\}_{j=H,F}, \lambda_t)$ from (3.4), (3.7), (3.8) and (B.7) into the arbitrageurs' first-order condition. Substituting into (3.10), we find an equation that is affine in q_t. Equation (B.1) follows by identifying the linear terms in q_t, and (B.2) follows by identifying the constant terms. Substituting into (3.11), we find an equation that is affine in q_t. Equation (B.3) follows by identifying the linear terms in q_t, and (B.4) follows by identifying the constant terms. The initial conditions $A_j(0) = C_j(0) = 0$ follow because the price of a bond with zero maturity is its face value, which is one.

Solving the system of (B.1)-(B.4) reduces to solving a system of 25 nonlinear scalar equations. Indeed, taking the 5×5 matrix M as given, we can solve the system (B.1) of five scalar equations in the elements of the 5×1 vector A_e, the system (B.3) of five linear ODEs in the elements of the 5×1 vector $A_H(\tau)$, and the same system (B.3) of five linear ODEs in the elements of the 5×1 vector $A_F(\tau)$. We can then substitute back into the definition (B.5) of M to derive the system of 25 nonlinear scalar equations. Given a solution to that system, (B.2) determines C_e uniquely, and (B.4) determines $(C_H(\tau), C_F(\tau))$ uniquely. Since for $a = 0$, (B.5) implies $M = \Gamma^\top$ and (B.6) implies $\lambda_C = 0$, (B.1) implies $A_e = (\Gamma^{-1})^\top (\mathcal{E}_{iH} - \mathcal{E}_{iF})$ and (B.2) implies (3.17). ∎

Corollary B.1 specializes Proposition B.1 to the case where arbitrageurs are risk-neutral or their risk aversion goes to zero.

Corollary B.1. *Suppose that arbitrage is global.*

- *When arbitrageurs are risk-neutral ($a = 0$), UIP and EH hold: the expected return on foreign currency is $\mu_{et}^{UIP} \equiv i_{Ht} - i_{Ft}$, and the expected return on country-j bonds is $\mu_{jt}^{(\tau)EH} \equiv i_{jt}$. Stationarity of the real exchange rate, as per the conjecture (3.1), requires (3.17).*
- *When arbitrageurs' risk aversion goes to zero ($a \to 0$), the expected return on foreign currency goes to μ_{et}^{UIP} and the expected return on country-j bonds goes to $\mu_{jt}^{(\tau)EH}$ only when (3.17) holds. Stationarity of the real exchange rate does not require (3.17) if $\alpha_e > 0$.*

Proof of Corollary B.1: The results for $a = 0$ follow from the arguments before the corollary' statement. When a goes to zero, (B.1), (B.3) and (B.5) imply that M goes to $\Gamma^\top$ and $(A_e, \{A_j(\tau)\}_{j=H,F})$ have the finite limits

$$\lim_{a\to 0} A_e = \left(\Gamma^{-1}\right)^\top (\mathcal{E}_{iH} - \mathcal{E}_{iF}),$$
$$\lim_{a\to 0} A_j(\tau) = \left(\Gamma^{-1}\right)^\top \left(I - e^{-\Gamma^\top \tau}\right) \mathcal{E}_{ij}.$$

When (3.17) holds, (B.2), (B.6) (B.6) are met with λ_C having a zero limit and $(C_e, \{C_j(\tau)\}_{j=H,F})$ having finite limits. Equation (3.16) then implies that λ_t goes to zero, which means from (3.10) and (3.11) that UIP and EH hold in the limit. When instead (3.17) does not hold, (B.2) implies that $A_e^\top \lambda_C$ has a non-zero limit. When, in addition, $\alpha_e > 0$, (B.2), (B.6) (B.6) are met with λ_C having a non-zero limit, $\{C_j(\tau)\}_{j=H,F}$ having finite limits, and C_e going to plus or minus infinity at the rate $\frac{1}{a}$. Equation (3.16) then implies that λ_t does not go to zero, which means from (3.10) and (3.11) that UIP and EH do not hold. ∎

Propositions B.2 and B.3 characterize the equilibrium in the currency and bond markets, respectively, under segmented arbitrage and the parameter restrictions assumed in Section 4.

Proposition B.2. *Suppose that arbitrage is segmented, the matrices* (Γ, Σ) *are diagonal, and* $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. *The exchange rate* e_t *is given by (3.1), with* (A_{iHe}, A_{iFe}) *positive and equal to the unique solution of*

$$\kappa_{ij} A_{ije} - 1 = -a_e \alpha_e A_{ije} \left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2 \right), \tag{B.8}$$

and C_e *solving*

$$\begin{aligned} & -\kappa_{iH} \bar{i}_H A_{iHe} + \kappa_{iF} \bar{i}_F A_{iFe} - (\pi_F - \pi_H) + \frac{1}{2} \sigma_{iH}^2 A_{iHe}^2 + \frac{1}{2} \sigma_{iF}^2 A_{iFe}^2 \\ & = a_e \left(\zeta_e - \alpha_e C_e \right) \left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2 \right). \end{aligned} \tag{B.9}$$

Proof of Proposition B.2: The first-order condition in the currency market is

$$\mu_{et} + i_{Ft} - i_{Ht} = A_e^\top \lambda_{et}, \tag{B.10}$$

where $A_e \equiv (A_{iHe}, -A_{iFe})$ and $\lambda_{et} \equiv a_e W_{Ft} (\sigma_{iH}^2 A_{iHe}, -\sigma_{iF}^2 A_{iFe})$. It follows from (3.10) by keeping only the term $W_{Ft} A_e$ in the parenthesis in (3.12), taking Σ to be diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$, and replacing a by a_e. Proceeding as in the derivation of (3.16) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we find $\lambda_{et} = (\lambda_{eHt}, \lambda_{eFt})$ with

$$\lambda_{ejt} = a_e \sigma_{ij}^2 \left[\zeta_e - \alpha_e \left(A_{iHe} i_{Ht} - A_{iFe} i_{Ft} + C_e \right) \right] A_{ije} (-1)^{1_{\{j=F\}}} \tag{B.11}$$

for $j = H, F$. Since (Γ, Σ) are diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$ and $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we can write (3.4) as

$$\mu_{et} = -A_{iHe} \kappa_{iH} (\bar{i}_H - i_{Ht}) + A_{iFe} \kappa_{iF} (\bar{i}_F - i_{Ft}) - (\pi_F - \pi_H) + \frac{1}{2} A_{iHe}^2 \sigma_{iH}^2 + \frac{1}{2} A_{iFe}^2 \sigma_{iF}^2. \tag{B.12}$$

Substituting λ_{et} from (B.11) and μ_{et} from (B.12) into (B.10), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equation (B.8) follows by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.9) follows by identifying the constant terms.

When $a\alpha_e = 0$, (B.8) has the unique solution $(A_{iHe}, A_{iFe}) = \left(\frac{1}{\kappa_{iH}}, \frac{1}{\kappa_{iF}} \right)$, which is positive. Consider next the case $a\alpha_e > 0$. A solution (A_{iHe}, A_{iFe}) to (B.8) must be positive, as can be seen by writing that equation as

$$\left[\kappa_{ij} + a_e \alpha_e \left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2 \right) \right] A_{ije} = 1. \tag{B.13}$$

Since $(A_{iHe}, A_{iFe}, a\alpha_e)$ are positive, the right-hand side of (B.8) is negative. Therefore, the left-hand side is negative as well, which implies $A_{iHe} < \frac{1}{\kappa_{Hj}}$ and $A_{iFe} < \frac{1}{\kappa_{Fj}}$. Dividing (B.8) written for $j = H$ by (B.8) written for $j = F$, we find

$$\frac{1 - \kappa_{iH} A_{iHe}}{1 - \kappa_{iF} A_{iFe}} = \frac{A_{iHe}}{A_{iFe}} \Leftrightarrow A_{iHe} = \frac{A_{iFe}}{1 + (\kappa_{iH} - \kappa_{iF}) A_{iFe}}. \tag{B.14}$$

Equation (B.14) determines A_{iHe} as an increasing function of $A_{iFe} \in \left[0, \frac{1}{\kappa_{iF}}\right]$, equal to zero for $A_{iFe} = 0$, and equal to $\frac{1}{\kappa_{iH}}$ for $A_{iFe} = \frac{1}{\kappa_{iF}}$. Substituting A_{iHe} as a function of A_{iFe} in (B.13) written for $j = F$, we find an equation in the single unknown A_{iFe}. The left-hand side of that equation is increasing in A_{iFe}, is equal to zero for $A_{iFe} = 0$, and is equal to a value larger than one for $A_{iFe} = \frac{1}{\kappa_{iF}}$. Hence, that equation has a unique solution $A_{iFe} \in \left(0, \frac{1}{\kappa_{iF}}\right)$. Given that solution, (B.14) determines $A_{iHe} \in \left(0, \frac{1}{\kappa_{iH}}\right)$ uniquely. Given (A_{iHe}, A_{iFe}), (B.9) determines C_e uniquely if $\alpha_e > 0$. If $\alpha_e = 0$, then the restriction

$$\bar{i}_H - \pi_H = \bar{i}_F - \pi_F + \left(\frac{1}{2} - a_e \zeta_e\right) \left(\frac{\sigma_{iH}^2}{\kappa_{iH}^2} + \frac{\sigma_{iF}^2}{\kappa_{iF}^2}\right) \tag{B.15}$$

on model parameters must be imposed and C_e is indeterminate. ∎

Proposition B.3. *Suppose that arbitrage is segmented, the matrices* (Γ, Σ) *are diagonal, and* $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. *Bond prices* $P_{jt}^{(\tau)}$ *in country* $j = H, F$ *are given by (3.2), with* $A_{ij'j}(\tau)$ *equal to zero for* $j' \neq j$ *and* $(A_{ijj}(\tau), C_{ij}(\tau))$ *equal to the unique solution of the system*

$$A'_{ijj}(\tau) + \kappa_{ij} A_{ijj}(\tau) - 1 = -a_j \sigma_{ij}^2 A_{ijj}(\tau) \int_0^T \alpha_j(\tau) A_{ijj}(\tau)^2 d\tau, \tag{B.16}$$

$$C'_j(\tau) - \kappa_{ij} \bar{i}_j A_{ijj}(\tau) + \frac{1}{2} \sigma_{ij}^2 A_{ijj}(\tau) \left(A_{ijj}(\tau) - 2A_{iFe} 1_{\{j=F\}}\right)$$
$$= a_j \sigma_{ij}^2 A_{ijj}(\tau) \int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) C_j(\tau)\right] A_{ijj}(\tau) d\tau, \tag{B.17}$$

with the initial conditions $A_{ijj}(0) = C_j(0) = 0$.

Proof of Proposition B.3: The first-order condition in the country-j bond market is

$$\mu_{jt}^{(\tau)} - i_{jt} = A_j(\tau) \lambda_{jt}, \tag{B.18}$$

where $A_j(\tau) \equiv A_{ijj}(\tau)$ and $\lambda_{jt} \equiv a_j \sigma_{ij}^2 \int_0^T X_{jt}^{(\tau)} A_{ijj}(\tau) d\tau$. It follows from (3.11) by keeping only the term $\int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau$ in the parenthesis in (3.12), taking Σ to be diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$, replacing a by a_j, and conjecturing that in equilibrium $A_{ij'j}(\tau) = 0$ for $j' \neq j$. Proceeding as in the derivation of (3.16) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$ and

$A_{ij'j}(\tau) = 0$ for $j' \neq j$, we find

$$\lambda_{jt} = a_j \sigma_{ij}^2 \left(\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) \left(A_{ijj}(\tau) i_{jt} + C_j(\tau) \right) \right] A_{ijj}(\tau) d\tau \right). \tag{B.19}$$

Since (Γ, Σ) are diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$, $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, and $A_{ij'j}(\tau) = 0$ for $j' \neq j$, we can write (3.7) and (3.8) as

$$\mu_{jt}^{(\tau)} = A'_{ijj}(\tau) i_{jt} + C'_j(\tau) - A_{ijj}(\tau) \kappa_{ij} (\bar{i}_j - i_{jt}) + A_{ijj}(\tau) \left(A_{ijj}(\tau) - 2 A_{iFe} 1_{\{j=F\}} \right) \sigma_{ij}^2. \tag{B.20}$$

Substituting λ_{jt} from (B.19) and μ_{jt} from (B.20) into (B.18), we find an equation that is affine in i_{jt}. Equation (B.16) follows by identifying the linear terms in i_{jt}, and (B.17) follows by identifying the constant terms. The initial conditions $A_{ijj}(0) = C_j(0) = 0$ follow because the price of a bond with zero maturity is its face value, which is one. Since the affine equation holds when (B.16) and (B.17) hold, our conjecture $A_{ij'j}(\tau) = 0$ for $j' \neq j$ is validated.

Solving (B.16) with the initial condition $A_{ijj}(0) = 0$, we find

$$A_{ijj}(\tau) = \frac{1 - e^{-\kappa_{ij}^* \tau}}{\kappa_{ij}^*}, \tag{B.21}$$

with

$$\kappa_{ij}^* \equiv \kappa_{ij} + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) A_{ijj}(\tau)^2 d\tau. \tag{B.22}$$

Substituting $A_{ijj}(\tau)$ from (B.21) into (B.22), we find the equation

$$\kappa_{ij}^* - \kappa_{ij} + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) \left(\frac{1 - e^{-\kappa_{ij}^* \tau}}{\kappa_{ij}^*} \right)^2 d\tau = 0 \tag{B.23}$$

in the single unknown κ_{ij}^*. The left-hand side of (B.23) is increasing in κ_{ij}^*, is negative for $\kappa_{ij}^* = \kappa_{ij}$, and goes to infinity when κ_{ij}^* goes to infinity. Hence, (B.23) has a unique solution $\kappa_{ij}^* > \kappa_{ij}$. Given κ_{ij}^*, (B.21) determines $A_{ijj}(\tau)$ uniquely.

Solving (B.17) with the initial condition $C_j(\tau) = 0$, we find

$$C_j(\tau) = \kappa_{ij}^* \bar{i}_j^* \int_0^\tau A_{ijj}(\tau) d\tau - \frac{1}{2} \sigma_{ij}^2 \int_0^\tau A_{ijj}(\tau)^2 d\tau, \tag{B.24}$$

with

$$\kappa_{ij}^* \bar{i}_j^* \equiv \kappa_{ij} \bar{i}_j + a_j \sigma_{ij}^2 \int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) C_j(\tau) \right] A_{ijj}(\tau) d\tau + \sigma_{ij}^2 A_{iFe} 1_{\{j=F\}}. \tag{B.25}$$

Substituting $C_j(\tau)$ from (B.24) into (B.25), we find

$$\bar{i}_j^* = \frac{\kappa_{ij}\bar{i}_j + a_j\sigma_{ij}^2 \int_0^T \zeta_j(\tau)A_{ijj}(\tau)d\tau + \sigma_{ij}^2 A_{iFe}1_{\{j=F\}} + \frac{1}{2}a_j\sigma_{ij}^4 \int_0^T \alpha_j(\tau)\left(\int_0^\tau A_{ijj}(\tau')^2 d\tau'\right)A_{ijj}(\tau)d\tau}{\kappa_{ij}^*\left[1 + a_j\sigma_{ij}^2 \int_0^T \alpha_j(\tau)\left(\int_0^\tau A_{ijj}(\tau')d\tau'\right)A_{ijj}(\tau)d\tau\right]}. \tag{B.26}$$

Given $\bar{i}_j^*$, (B.24) determines $C_j(\tau)$ uniquely. ∎

Proof of Proposition 4.1: The property $A_{ije} > 0$ is shown in the proof of Proposition B.2. The UIP value of A_{ije} is $A_{ije}^{UIP} \equiv \frac{1}{\kappa_{ij}}$, as can be seen from (B.8) by setting $a_e = 0$. When $a_e > 0$ and $\alpha_e > 0$, the proof of Proposition B.2 shows $A_{ije} < \frac{1}{\kappa_{ij}}$. Differentiating (B.12) with respect to i_{Ht} and i_{Ft}, we find

$$\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ht}} = \kappa_{iH}A_{iHe} - 1 < 0,$$
$$\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ft}} = -\kappa_{iF}A_{iFe} + 1 > 0,$$

respectively. ∎

Proof of Proposition 4.2: The properties $A_{ijj}(\tau) > 0$ and $A_{ij'j} = 0$ for $j' \neq j$ are shown in the proof of Proposition B.3. The EH value of $A_{ijj}(\tau)$ is $A_{ijj}^{EH}(\tau) \equiv \frac{1-e^{-\kappa_{ij}\tau}}{\kappa_{ij}}$, as can be seen from (B.21) and (B.22) by setting $a_j = 0$. When $a_j > 0$ and $\alpha_j(\tau) > 0$, (B.22) implies $\kappa_{ij}^* > \kappa_{ij}$ and (B.21) implies $A_{ijj}(\tau) < A_{ijj}^{EH}(\tau)$. Differentiating (B.20) with respect to i_{jt}, we find

$$\frac{\partial\left(\mu_{jt}^{(\tau)} - i_{jt}\right)}{\partial i_{jt}} = A'_{ijj}(\tau) + \kappa_{ij}A_{ijj}(\tau) - 1 = (\kappa_{ij} - \kappa_{ij}^*)A_{ijj}(\tau) < 0,$$

where the second step follows from (B.21). ∎

Proof of Proposition 4.3: Consider an one-off increase in γ_t at time zero, and denote by κ_γ the rate at which γ_t reverts to its mean of zero. Equation (B.11) is modified to

$$\lambda_{ejt} = a_e\sigma_{ij}^2\left[\zeta_e + \theta_e\gamma_t - \alpha_e\left(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}e_t + C_e\right)\right]A_{ijt}(-1)^{1_{\{j=F\}}} \tag{B.27}$$

and (B.12) is modified to

$$\mu_{et} = -A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\gamma e}\kappa_\gamma\gamma_t - (\pi_F - \pi_H) + \frac{1}{2}A_{iHe}^2\sigma_{iH}^2 + \frac{1}{2}A_{iFe}^2\sigma_{iF}^2. \tag{B.28}$$

Substituting λ_{ejt} from (B.27) and μ_{et} from (B.28) into (B.10), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$\begin{aligned}
&\kappa_\gamma A_{\gamma e} = a_e(\theta_e - \alpha_e A_{\gamma e}) \left(A_{iHe}^2 \sigma_{iH}^2 + A_{iFe}^2 \sigma_{iF}^2\right) \\
&\Rightarrow A_{\gamma e} = \frac{a_e \theta_e \left(A_{iHe}^2 \sigma_{iH}^2 + A_{iFe}^2 \sigma_{iF}^2\right)}{\kappa_\gamma + a_e \alpha_e \left(A_{iHe}^2 \sigma_{iH}^2 + A_{iFe}^2 \sigma_{iF}^2\right)}.
\end{aligned} \tag{B.29}$$

When $\alpha_e > 0$, (B.29) implies $A_{\gamma e} > 0$ because $\theta_e > 0$. Hence, an increase in γ_t causes the foreign currency to depreciate. Since bonds in each country are traded by a separate set of agents than those trading foreign currency, their prices do not depend on γ_t.

Consider next an one-off increase in β_{jt} at time zero, and denote by $\kappa_{\beta j}$ the rate at which β_{jt} reverts to its mean of zero. Equation (B.19) is modified to

$$\lambda_{jt} = a_j \sigma_{ij}^2 \left(\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) \left(A_{ijj}(\tau) i_{jt} + A_{\beta jj}(\tau) \beta_{jt} + C_j(\tau)\right)\right] A_{ijj}(\tau) d\tau \right), \tag{B.30}$$

and (B.20) is modified to

$$\begin{aligned}
\mu_{jt}^{(\tau)} =& A'_{ijj}(\tau) i_{jt} + A'_{\beta jj}(\tau) \beta_{jt} + C'_j(\tau) - A_{ijj}(\tau) \kappa_{ij} (\bar{i}_j - i_{jt}) + A_{\beta jj}(\tau) \kappa_{\beta j} \beta_{jt} \\
&+ \frac{1}{2} A_{ijj}(\tau) \left(A_{ijj}(\tau) - 2 A_{iFe} 1_{\{j=F\}}\right) \sigma_{ij}^2.
\end{aligned} \tag{B.31}$$

Substituting λ_{jt} from (B.30) and μ_{jt} from (B.31) into (B.18), we find an equation that is affine in (i_{jt}, β_{jt}). Identifying the linear terms in β_{jt} yields

$$A'_{\beta jj}(\tau) + \kappa_{\beta j} A_{\beta jj}(\tau) = a_j \sigma_{ij}^2 A_{ijj}(\tau) \int_0^T \left[\theta_j(\tau) - \alpha_j(\tau) A_{\beta jj}(\tau)\right] A_{ijj}(\tau) d\tau. \tag{B.32}$$

Solving (B.32) with the initial condition $A_{\beta jj}(\tau) = 0$, we find

$$A_{\beta jj}(\tau) = \bar{\lambda}_{ij\beta j} \int_0^\tau A_{ijj}(\tau') e^{-\kappa_{\beta j}(\tau - \tau')} d\tau', \tag{B.33}$$

where

$$\bar{\lambda}_{ij\beta j} \equiv a_j \sigma_{ij}^2 \int_0^T \left[\theta_j(\tau) - \alpha_j(\tau) A_{\beta jj}(\tau)\right] A_{ijj}(\tau) d\tau. \tag{B.34}$$

Substituting $A_{\beta jj}(\tau)$ from (B.33) into (B.34) and solving for $\bar{\lambda}_{ij\beta j}$, we find

$$\bar{\lambda}_{ij\beta j} = \frac{a_j \sigma_{ij}^2 \int_0^T \theta_j(\tau) A_{ijj}(\tau) d\tau}{1 + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) \left(\int_0^\tau A_{ijj}(\tau') e^{-\kappa_{\beta j}(\tau - \tau')} d\tau'\right) A_{ijj}(\tau) d\tau}. \tag{B.35}$$

When $a_j > 0$, (B.35) implies $\lambda_{ij\beta j} > 0$ and (B.33) implies $A_{\beta jj}(\tau) > 0$ because $(\theta_j(\tau), A_{ijj}(\tau))$ are positive. Hence, an increase in β_{jt} raises bond yields in country j. Since the foreign currency and bonds in country j' are traded by different agents than those trading bonds in country j, their prices do not depend on β_{jt}. ∎

Proposition B.4 characterizes the equilibrium under global arbitrage and the parameter restrictions assumed in Section 4.

Proposition B.4. *Suppose that arbitrage is global, the matrices (Γ, Σ) are diagonal, and $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. The exchange rate e_t is given by (3.1) and bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (3.2), with $(\{A_{ije}\}_{j=H,F}, C_e)$ solving*

$$\kappa_{ij} A_{ije} - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ije} - a\sigma_{ij'}^2 \bar{\lambda}_{ij'j} A_{ij'e}, \tag{B.36}$$

$$\begin{aligned} & -\kappa_{iH}\bar{i}_H A_{iHe} + \kappa_{iF}\bar{i}_F A_{iFe} - (\pi_F - \pi_H) + \frac{1}{2}\sigma_{iH}^2 A_{iHe}^2 + \frac{1}{2}\sigma_{iF}^2 A_{iFe}^2 \\ & = a\sigma_{iH}^2 \bar{\lambda}_{iHC} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{iFC} A_{iFe}, \end{aligned} \tag{B.37}$$

and $(A_{ijj}(\tau), A_{ijj'}(\tau), C_j(\tau))$ solving

$$A'_{ijj}(\tau) + \kappa_{ij} A_{ijj}(\tau) - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j} A_{ij'j}(\tau), \tag{B.38}$$

$$A'_{ij'j}(\tau) + \kappa_{ij'} A_{ij'j}(\tau) = a\sigma_{ij}^2 \bar{\lambda}_{ijj'} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j'} A_{ij'j}(\tau), \tag{B.39}$$

$$\begin{aligned} & C'_j(\tau) - \kappa_{ij}\bar{i}_j A_{ijj}(\tau) - \kappa_{ij'}\bar{i}_{j'} A_{ij'j}(\tau) + \frac{1}{2}\sigma_{ij}^2 A_{ijj}(\tau)\left(A_{ijj}(\tau) - 2A_{iFe}1_{\{j=F\}}\right) \\ & + \frac{1}{2}\sigma_{ij'}^2 A_{ij'j}(\tau)\left(A_{ij'j}(\tau) + 2A_{iHe}1_{\{j=F\}}\right) = a\sigma_{ij}^2 \bar{\lambda}_{ijC} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'C} A_{ij'j}(\tau), \end{aligned} \tag{B.40}$$

with the initial conditions $A_{ijj}(0) = A_{ijj'}(0) = C_j(0) = 0$, where $j' \neq j$ and

$$\bar{\lambda}_{ijj} \equiv -\alpha_e A_{ije}^2 - \sum_{k=H,F} \int_0^T \alpha_k(\tau) A_{ijk}(\tau)^2 d\tau, \tag{B.41}$$

$$\bar{\lambda}_{ijj'} \equiv \alpha_e A_{ije} A_{ij'e} - \sum_{k=H,F} \int_0^T \alpha_k(\tau) A_{ijk}(\tau) A_{ij'k}(\tau) d\tau, \tag{B.42}$$

$$\bar{\lambda}_{ijC} \equiv (\zeta_e - \alpha_e C_e) A_{ije}(-1)^{1_{\{j=F\}}} + \sum_{k=H,F} \int_0^T (\zeta_k(\tau) - \alpha_k(\tau) C_k(\tau)) A_{ijk}(\tau) d\tau. \tag{B.43}$$

Proof of Proposition B.4: The first-order conditions are (3.10) and (3.11), with $A_e \equiv (A_{iHe}, -A_{iFe})$, $A_j(\tau) \equiv (A_{iHj}(\tau), A_{iFj}(\tau))$ and $\lambda_t \equiv (\lambda_{iHt}, \lambda_{iFt})$ with

$$\lambda_{ijt} \equiv a\sigma_{ij}^2 \left(W_{Ft} A_{ije}(-1)^{1_{\{j=F\}}} + \sum_{j'=H,F} \int_0^T X_{j't}^{(\tau)} A_{ijj'}(\tau) d\tau \right). \tag{B.44}$$

They follow from (3.10) and (3.11) by taking Σ to be diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$.

Proceeding as in the derivation of (3.16) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we find

$$\begin{aligned}
\lambda_{ijt} = a\sigma_{ij}^2 \Bigg(& \left[\zeta_e - \alpha_e \left(A_{iHe} i_{Ht} - A_{iFe} i_{Ft} + C_e\right)\right] A_{ije} (-1)^{1_{\{j=F\}}} \\
& + \sum_{j'=H,F} \int_0^T \left[\zeta_{j'}(\tau) - \alpha_{j'}(\tau) \left(A_{iHj'}(\tau) i_{Ht} + A_{iFj'}(\tau) i_{Ft} + C_{j'}(\tau)\right)\right] A_{ijj'}(\tau) d\tau \Bigg) \\
\equiv a\sigma_{ij}^2 & \left(\bar{\lambda}_{ijj} i_{jt} + \bar{\lambda}_{ijj'} i_{j't} + \bar{\lambda}_{ijC}\right). \qquad \text{(B.45)}
\end{aligned}$$

Since (Γ, Σ) are diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$ and $\beta_{Ht} = \beta_{Ft} = \gamma_t = 0$, we can write (3.7) and (3.8) as

$$\begin{aligned}
\mu_{jt}^{(\tau)} \equiv & A'_{iHj}(\tau) i_{Ht} + A'_{iFj}(\tau) i_{Ft} + C'_j(\tau) - A_{iHj}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) - A_{iFj}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft}) \\
& + \frac{1}{2} A_{iHj}(\tau) \left(A_{iHj}(\tau) + 2A_{iHe} 1_{\{j=F\}}\right) \sigma_{iH}^2 + \frac{1}{2} A_{iFj}(\tau) \left(A_{iFj}(\tau) - 1_{\{j=F\}} 2A_{iFe}\right) \sigma_{iF}^2. \qquad \text{(B.46)}
\end{aligned}$$

Substituting λ_t from (B.45) and μ_{et} from (B.12) into (3.10) (for the definitions of (A_e, λ_t) in Section 4.2), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equation (B.36) follows by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.37) follows by identifying the constant terms. Substituting λ_t from (B.45) and $\mu_{jt}^{(\tau)}$ from (B.46) into (3.11) (for the definitions of $(A_j(\tau), \lambda_t)$ in Section 4.2), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equations (B.38) and (B.39) follow by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.40) follows by identifying the constant terms. The initial conditions $A_{ijj}(0) = A_{ijj'}(0) = C_j(0) = 0$ follow because the price of a bond with zero maturity is its face value, which is one.

Solving the system of (B.36)-(B.43) reduces to solving a system of three nonlinear scalar equations. Indeed, taking $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$ as given, we can solve the system (B.36) of two scalar equations in (A_{iHe}, A_{iFe}), the system (B.38) and (B.39) of two linear ODEs in $(A_{iHH}(\tau), A_{iFH}(\tau))$ (setting $(j, j') = (H, F)$), and the same system (B.38) and (B.39) of two linear ODEs in $(A_{iHF}(\tau), A_{iFF}(\tau))$ (setting $(j, j') = (F, H)$). We can then substitute back into the definitions of $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$ to derive the system of three nonlinear scalar equations. Given a solution of that system, (B.40) determines $(C_H(\tau), C_F(\tau))$ uniquely, and (B.37) determines C_e uniquely if $\alpha_e > 0$. If $\alpha_e = 0$, then a parameter restriction analogous to (B.15) must be imposed and C_e is indeterminate. The results in Section 4.2 hold for any solution $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$. ∎

Proof of Proposition 4.4: We start by proving a series of lemmas.

Lemma B.1. *The matrix*

$$M_i \equiv \begin{pmatrix} \kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH} & -a\sigma_{iF}^2\bar{\lambda}_{iFH} \\ -a\sigma_{iH}^2\bar{\lambda}_{iHF} & \kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF} \end{pmatrix} \tag{B.47}$$

has two positive eigenvalues.

Proof: The characteristic polynomial of M_i is

$$\Pi(\lambda) \equiv \left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH} - \lambda\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF} - \lambda\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}. \tag{B.48}$$

For $\lambda = 0$, $\Pi(\lambda)$ takes the value

$$\begin{aligned}
\Pi(0) &= \left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH} \\
&> a^2\sigma_{iH}^2\sigma_{iH}^2\left(\bar{\lambda}_{iHH}\bar{\lambda}_{iFF} - \bar{\lambda}_{iHF}\bar{\lambda}_{iFH}\right) \\
&= a^2\sigma_{iH}^2\sigma_{iH}^2\Bigg[\left(\alpha_e A_{iHe}^2 + \int_0^T \alpha_H(\tau)A_{iHH}(\tau)^2 d\tau + \int_0^T \alpha_F(\tau)A_{iHF}(\tau)^2 d\tau\right) \\
&\quad \times \left(\alpha_e A_{iFe}^2 + \int_0^T \alpha_H(\tau)A_{iFH}(\tau)^2 d\tau + \int_0^T \alpha_F(\tau)A_{iFF}(\tau)^2 d\tau\right) \\
&\quad - \left(\alpha_e A_{iHe}A_{iFe} - \int_0^T \alpha_H(\tau)A_{iHH}(\tau)A_{iFH}(\tau)d\tau + \int_0^T \alpha_F(\tau)A_{iHF}(\tau)A_{iFF}(\tau)d\tau\right)^2\Bigg].
\end{aligned} \tag{B.49}$$

The second step in (B.49) follows because $(\kappa_{iH}, \kappa_{iF})$ are positive and because (B.41) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive. The third step in (B.49) follows from (B.41) and (B.42). The Cauchy-Schwarz inequality associated to the scalar product

$$X \cdot Y \equiv \alpha_e xy + \int_0^T \alpha_H(\tau)X_H(\tau)Y_H(\tau)d\tau + \int_0^T \alpha_F(\tau)X_F(\tau)Y_F(\tau)d\tau$$

where $X \equiv (x, X_H(\tau), X_F(\tau))$, $Y \equiv (y, Y_H(\tau), Y_F(\tau))$, (x, y) are scalars, and $(X_H(\tau), X_F(\tau), Y_H(\tau), Y_F(\tau))$ are functions of τ, implies that (B.49) is non-negative. Hence, $\Pi(0) > 0$.

For $\lambda = \kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}$ and $\lambda = \kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}$, $\Pi(\lambda)$ takes the value $-a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}$, which is non-positive because (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$. Since $(\kappa_{iH}, \kappa_{iF})$ are positive and $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive, $\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}$ and $\lambda = \kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}$ are positive. Since $\Pi(\lambda)$ is a quadratic function of λ, is positive for $\lambda = 0$, is non-positive for two positive values of λ, and goes to infinity when λ goes to infinity, it has two positive roots. ∎

The matrix M_i plays an important role in the determination of (A_{iHe}, A_{iFe}) and $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$. Equation (B.36) gives rise to the linear system

$$M_i \begin{pmatrix} A_{iHe} \\ A_{iFe} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{B.50}$$

Since M_i has two positive eigenvalues, it is invertible, and hence (B.50) can be solved for (A_{iHe}, A_{iFe}). Equations (B.38) and (B.39) give rise to the linear system

$$\begin{pmatrix} A_{iHH}(\tau) \\ A_{iFH}(\tau) \end{pmatrix}' + M_i \begin{pmatrix} A_{iHH}(\tau) \\ A_{iFH}(\tau) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B.51}$$

for $(j, j') = (H, F)$, and to

$$\begin{pmatrix} A_{iHF}(\tau) \\ A_{iFF}(\tau) \end{pmatrix}' + M_i \begin{pmatrix} A_{iHF}(\tau) \\ A_{iFF}(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{B.52}$$

for $(j, j') = (F, H)$. Since M_i has two positive eigenvalues, the solutions $(A_{iHH}(\tau), A_{iFH}(\tau))$ to (B.51) and $(A_{iHF}(\tau), A_{iFF}(\tau))$ to (B.52) go to finite limits when τ goes to infinity.

Lemma B.2. *The normalized factor prices $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are non-negative.*

Proof: Suppose, proceeding by contradiction, that $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are negative. The solution to (B.50) is

$$A_{iHe} = \frac{\kappa_{iF} - a\sigma_{iF}^2(\bar{\lambda}_{iFF} + \bar{\lambda}_{iFH})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}}, \tag{B.53}$$

$$A_{iFe} = \frac{\kappa_{iH} - a\sigma_{iH}^2(\bar{\lambda}_{iHH} + \bar{\lambda}_{iHF})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}}. \tag{B.54}$$

The denominator in (B.53) and (B.54) is $\Pi(0) > 0$. The numerators in (B.53) and (B.54) are positive because $(\kappa_{iH}, \kappa_{iF})$ are positive and $(a\bar{\lambda}_{iHH}, a\bar{\lambda}_{iFF}, a\bar{\lambda}_{iHF}, a\bar{\lambda}_{iFH})$ are non-positive. Hence, (A_{iHe}, A_{iFe}) are positive.

When $a = 0$, (B.39) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$ for all $\tau > 0$. Since, in addition, $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \geq 0$, a contradiction.

When $a > 0$, (B.38) and (B.39) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Moreover, differentiating (B.39), we find $A''_{iFH}(0) = a\sigma_{iH}^2\bar{\lambda}_{iHF}A'_{iHH}(0) < 0$ and $A''_{iHF}(0) = a\sigma_{iF}^2\bar{\lambda}_{iFH}A'_{iFF}(0) < 0$. Hence, $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) < 0$ and $A_{iFH}(\tau) < 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{iHH}(\tau') > 0,\ A_{iFF}(\tau') > 0,\ A_{iHF}(\tau') < 0 \text{ and } A_{iFH}(\tau') < 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is finite, then (i) $A_{iHH}(\tau_0) = 0$, $A'_{iHH}(\tau_0) \leq 0$, $A_{iFF}(\tau_0) \geq 0$, $A_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (ii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) = 0$, $A'_{iFF}(\tau_0) \leq 0$, $A_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (iii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (iv) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) < 0$, $A_{iFH}(\tau_0) = 0$ and $A'_{iFH}(\tau_0) \geq 0$. Case (i) yields a contradiction because (B.38) for $(j,j') = (H,F)$, $A_{iHH}(\tau_0) = 0$, $A_{iFH}(\tau_0) \leq 0$ and $\bar{\lambda}_{iFH} < 0$ imply $A'_{iHH}(\tau_0) \geq 1$. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.39) for $(j,j') = (H,F)$, $A_{iHH}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} < 0$ imply $A'_{iFH}(\tau_0) < 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) < 0$ and $A_{iFH}(\tau) < 0$ for all $\tau > 0$. Since, in addition, $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \geq 0$, a contradiction. Therefore, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are non-negative. ∎

Lemma B.3. *The functions $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are positive for all $\tau > 0$.*

- *When $a > 0$ and $\alpha_e > 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are positive for all $\tau > 0$.*
- *When $a = 0$ or $\alpha_e = 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are zero.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. If $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, then (B.39) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$ for all $\tau > 0$. Since, in addition, (B.53) and (B.54) imply $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$, a contradiction. Hence, Lemma B.2 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$.

Equations (B.38) and (B.39) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Moreover, differentiating (B.39), we find $A''_{iFH}(0) = a\sigma^2_{iH}\bar{\lambda}_{iHF}A'_{iHH}(0) > 0$ and $A''_{iHF}(0) = a\sigma^2_{iF}\bar{\lambda}_{iFH}A'_{iFF}(0) > 0$. Hence, $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{iHH}(\tau') > 0,\ A_{iFF}(\tau') > 0,\ A_{iHF}(\tau') > 0 \text{ and } A_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is finite, then (i) $A_{iHH}(\tau_0) = 0$, $A'_{iHH}(\tau_0) \leq 0$, $A_{iFF}(\tau_0) \geq 0$, $A_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (ii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) = 0$, $A'_{iFF}(\tau_0) \leq 0$, $A_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (iii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$, $A'_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (iv) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $A'_{iFH}(\tau_0) \leq 0$. Case (i) yields a contradiction because (B.38) for $(j,j') = (H,F)$, $A_{iHH}(\tau_0) = 0$, $A_{iFH}(\tau_0) \geq 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iHH}(\tau_0) \geq 1$. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.39) for $(j,j') = (H,F)$, $A_{iHH}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$

and $\bar{\lambda}_{iHF} > 0$ imply $A'_{iFH}(\tau_0) > 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$.

Consider next the case $a = 0$. The properties of $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ follow because (B.38) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies $A_{iHH}(\tau) = A^{EH}_{iHH}(\tau) \equiv \frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}} > 0$ and $A_{iFF}(\tau) = A^{EH}_{iFF}(\tau) \equiv \frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}} > 0$, and (B.39) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$.

Consider finally the case $a > 0$ and $\alpha_e = 0$. Suppose, proceeding by contradiction, that $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are positive. The argument in the case $a > 0$ and $\alpha_e > 0$ implies that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$. Since $\alpha_e = 0$, (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \leq 0$, a contradiction. Hence, Lemma B.2 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$. Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, (B.39) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Since $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$, (B.38) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies that $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive for all $\tau > 0$. ∎

Lemma B.4. *The functions $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are increasing. When $a > 0$ and $\alpha_e > 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are also increasing.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. Equations $A'_{iHH}(0) = A'_{iFF}(0) = 1$, $A'_{iHF}(0) = A'_{iFH}(0) = 0$, $A''_{iFH}(0) = a\sigma^2_{iH}\bar{\lambda}_{iHF}A'_{iHH}(0) > 0$ and $A''_{iHF}(0) = a\sigma^2_{iF}\bar{\lambda}_{iFH}A'_{iFF}(0) > 0$ imply $A'_{iHH}(\tau) > 0$, $A'_{iFF}(\tau) > 0$, $A'_{iHF}(\tau) > 0$ and $A'_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') > 0,\ A'_{iFF}(\tau') > 0,\ A'_{iHF}(\tau') > 0 \text{ and } A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is finite, then (i) $A'_{iHH}(\tau_0) = 0$, $A''_{iHH}(\tau_0) \leq 0$, $A'_{iFF}(\tau_0) \geq 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A'_{iFH}(\tau_0) \geq 0$, or (ii) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) = 0$, $A''_{iFF}(\tau_0) \leq 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0)' \geq 0$, or (iii) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) > 0$, $A'_{iHF}(\tau_0) = 0$, $A''_{iHF}(\tau_0) \leq 0$ and $A'_{iFH}(\tau_0) \geq 0$, or (iv) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) > 0$, $A'_{iHF}(\tau_0) > 0$, $A'_{iFH}(\tau_0) = 0$ and $A''_{iFH}(\tau_0) \leq 0$. To analyze Cases (i)-(iv), we use

$$A''_{ijj}(\tau) + \kappa_{ij}A'_{ijj}(\tau) = a\sigma^2_{ij}\bar{\lambda}_{ijj}A'_{ijj}(\tau) + a\sigma^2_{ij'}\bar{\lambda}_{ij'j}A'_{ij'j}(\tau), \tag{B.55}$$

$$A''_{ij'j}(\tau) + \kappa_{ij'}A'_{ij'j}(\tau) = a\sigma^2_{ij}\bar{\lambda}_{ijj'}A'_{ijj}(\tau) + a\sigma^2_{ij'}\bar{\lambda}_{ij'j'}A'_{ij'j}(\tau), \tag{B.56}$$

which follow from differentiating (B.38) and (B.39), respectively.

Case (i) yields a contradiction. Indeed, if $A''_{iHH}(\tau_0) = 0$, then (B.55) for $(j, j') = (H, F)$, $A'_{iHH}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iFH}(\tau_0) = 0$. The unique solution to the linear system of ODEs (B.55) and (B.56) for $(j, j') = (H, F)$ with the initial condition $(A'_{iHH}(\tau_0), A'_{iFH}(\tau_0)) = (0, 0)$ is the function that equals (0,0) for all τ. This yields a contradiction because $(A'_{iHH}(0), A'_{iFH}(0)) = (1, 0)$. Hence, $A''_{iHH}(\tau_0) < 0$, which combined with (B.55) for $(j,j') = (H,F)$, $A'_{iHH}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFH}(\tau_0) < 0$, again a contradiction. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.56) for $(j,j') = (H,F)$, $A'_{iHH}(\tau_0) > 0$, $A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} > 0$ imply $A''_{iFH}(\tau_0) > 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ are increasing.

In the case $a = 0$ or $\alpha_e = 0$, Lemma B.3 implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Since $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$, (B.38) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies that $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are increasing. ∎

Lemma B.5. *The scalars A_{iHe} and A_{iFe} are positive.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$ and $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$ (Lemma B.3), (B.42) implies $A_{iHe}A_{iFe} > 0$. Hence, (A_{iHe}, A_{iFe}) are either both positive or both negative. Suppose, proceeding by contradiction, that they are both negative. Equations (B.53) and (B.54) imply

$$\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH} < a\sigma_{iH}^2\bar{\lambda}_{iHF}, \tag{B.57}$$
$$\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF} < a\sigma_{iF}^2\bar{\lambda}_{iFH}. \tag{B.58}$$

Since the left-hand side in each of (B.57) and (B.58) is positive, (B.57) and (B.58) imply

$$\Pi(0) = \left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH} < 0,$$

a contradiction. Hence, (A_{iHe}, A_{iFe}) are positive.

Consider next the case $a = 0$. Equation (B.36) implies $A_{iHe} = A_{iHe}^{UIP} \equiv \frac{1}{\kappa_{Hj}} > 0$ and $A_{iFe} = A_{iFe}^{UIP} \equiv \frac{1}{\kappa_{Fj}} > 0$. Consider finally the case $\alpha_e = 0$ and $a > 0$. Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$ and $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive, (B.53) and (B.54) imply that (A_{iHe}, A_{iFe}) are positive. ∎

Lemma B.6. *The functions $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are positive for all $\tau > 0$.*

Proof: In the case $a = 0$ or $\alpha_e = 0$, the lemma follows from Lemma B.3. To prove the lemma in the case $a > 0$ and $\alpha_e > 0$, we proceed in two steps. In Step 1, we show that $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are positive in the limit when τ goes to infinity. In Step 2, we show that $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are either increasing in τ, or increasing and then decreasing. The lemma follows by combining these properties with $A_{iHH}(0) - A_{iHF}(0) = A_{iFF}(0) - A_{iFH}(0) = 0$.

Step 1: Limit at infinity. Since the matrix M has two positive eigenvalues, the functions $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ go to finite limits when τ goes to infinity. These limits solve the system of equations

$$\kappa_{ij}A_{ijj}(\infty) - 1 = a\sigma_{ij}^2\bar{\lambda}_{ijj}A_{ijj}(\infty) + a\sigma_{ij'}^2\bar{\lambda}_{ij'j}A_{ij'j}(\infty), \tag{B.59}$$

$$\kappa_{ij'}A_{ij'j}(\infty) = a\sigma_{ij}^2\bar{\lambda}_{ijj'}A_{ijj}(\infty) + a\sigma_{ij'}^2\bar{\lambda}_{ij'j'}A_{ij'j}(\infty), \tag{B.60}$$

which are derived from (B.38) and (B.39) by setting the derivatives to zero. Subtracting (B.60) for $(j, j') = (F, H)$ from (B.59) for $(j, j') = (H, F)$, we find

$$\begin{aligned}&\kappa_{iH}(A_{iHH}(\infty) - A_{iHF}(\infty)) - 1\\ &= a\sigma_{iH}^2\bar{\lambda}_{iHH}(A_{iHH}(\infty) - A_{iHF}(\infty)) + a\sigma_{iF}^2\bar{\lambda}_{iFH}(A_{iFH}(\infty) - A_{iFF}(\infty)).\end{aligned} \tag{B.61}$$

Subtracting (B.60) for $(j, j') = (H, F)$ from (B.59) for $(j, j') = (F, H)$, we similarly find

$$\begin{aligned}&\kappa_{iF}(A_{iFF}(\infty) - A_{iFH}(\infty)) - 1\\ &= a\sigma_{iH}^2\bar{\lambda}_{iHF}(A_{iHF}(\infty) - A_{iHH}(\infty)) + a\sigma_{iF}^2\bar{\lambda}_{iFF}(A_{iFF}(\infty) - A_{iFH}(\infty)).\end{aligned} \tag{B.62}$$

The solution to the system of (B.61) and (B.62) is

$$A_{iHH}(\infty) - A_{iHF}(\infty) = \frac{\kappa_{iF} - a\sigma_{iF}^2(\bar{\lambda}_{iFF} + \bar{\lambda}_{iFH})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}} = A_{iHe}, \tag{B.63}$$

$$A_{iFF}(\infty) - A_{iFH}(\infty) = \frac{\kappa_{iH} - a\sigma_{iH}^2(\bar{\lambda}_{iHH} + \bar{\lambda}_{iHF})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}} = A_{iFe}, \tag{B.64}$$

where the second equality in (B.63) and (B.64) follows from (B.53) and (B.54), respectively. Since (A_{iHe}, A_{iFe}) are positive (Lemma B.5), so are $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty))$.

Step 2: Monotonicity. Equations (B.38) and (B.39) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1 > 0$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Hence, $A'_{iHH}(\tau) - A'_{iHF}(\tau) > 0$ and $A'_{iFF}(\tau) - A'_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') - A'_{iHF}(\tau') > 0 \text{ and } A'_{iFF}(\tau') - A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is infinity, then $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are increasing in τ. Suppose instead that τ_0 is finite. Then, either (i) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$, $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) \leq 0$ and $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) \geq 0$, or (ii) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) > 0$, $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$

and $A''_{iFF}(\tau_0) - A''_{iFH}(\tau_0) \leq 0$. To analyze Cases (i) and (ii), we use

$$\begin{aligned} &A'_{iHH}(\tau) - A'_{iHF}(\tau) + \kappa_{iH}(A_{iHH}(\tau) - A_{iHF}(\tau)) - 1 \\ &= a\sigma^2_{iH}\bar{\lambda}_{iHH}(A_{iHH}(\tau) - A_{iHF}(\tau)) + a\sigma^2_{iF}\bar{\lambda}_{iFH}(A_{iFH}(\tau) - A_{iFF}(\tau)), \end{aligned} \tag{B.65}$$

which follows by subtracting (B.39) for $(j, j') = (F, H)$ from (B.59) for $(j, j') = (H, F)$, and

$$\begin{aligned} &A'_{iFF}(\tau) - A'_{iFH}(\tau) + \kappa_{iF}(A_{iFF}(\tau) - A_{iFH}(\tau)) - 1 \\ &= a\sigma^2_{iH}\bar{\lambda}_{iHF}(A_{iHF}(\tau) - A_{iHH}(\tau)) + a\sigma^2_{iF}\bar{\lambda}_{iFF}(A_{iFF}(\tau) - A_{iFH}(\tau)), \end{aligned} \tag{B.66}$$

which follows by subtracting (B.60) for $(j, j') = (H, F)$ from (B.59) for $(j, j') = (F, H)$. Differentiating (B.65) and (B.66), we find

$$\begin{aligned} &A''_{iHH}(\tau) - A''_{iHF}(\tau) + \kappa_{iH}(A'_{iHH}(\tau) - A'_{iHF}(\tau)) \\ &= a\sigma^2_{iH}\bar{\lambda}_{iHH}(A'_{iHH}(\tau) - A'_{iHF}(\tau)) + a\sigma^2_{iF}\bar{\lambda}_{iFH}(A'_{iFH}(\tau) - A'_{iFF}(\tau)) \end{aligned} \tag{B.67}$$

and

$$\begin{aligned} &A''_{iFF}(\tau) - A''_{iFH}(\tau) + \kappa_{iF}(A'_{iFF}(\tau) - A'_{iFH}(\tau)) \\ &= a\sigma^2_{iH}\bar{\lambda}_{iHF}(A'_{iHF}(\tau) - A'_{iHH}(\tau)) + a\sigma^2_{iF}\bar{\lambda}_{iFF}(A'_{iFF}(\tau) - A'_{iFH}(\tau)), \end{aligned} \tag{B.68}$$

respectively. Equations (B.67) and (B.68) are a linear system of ODEs in the functions $(A'_{iHH}(\tau) - A'_{iHF}(\tau), A'_{iFF}(\tau) - A'_{iFH}(\tau))$.

Consider first Case (i). If $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) = 0$, then (B.67), $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$. The unique solution to the linear system of ODEs (B.67) and (B.68) with the initial condition $(A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0), A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0)) = (0, 0)$ is the function that equals (0,0) for all τ. This yields a contradiction because $(A'_{iHH}(0) - A'_{iHF}(0), A'_{iFF}(0) - A'_{iFH}(0)) = (1, 1)$. Hence, $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) < 0$, which combined with (B.67), $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) > 0$. Since $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) < 0$, $A'_{iHH}(\tau) - A'_{iHF}(\tau) < 0$ for τ larger than and close to τ_0. We define τ'_0 by

$$\tau'_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') - A'_{iHF}(\tau') < 0 \text{ and } A'_{iFF}(\tau') - A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (\tau_0, \tau)\}.$$

If τ'_0 is finite, then either (ia) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$, $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) \geq 0$ and $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) \geq 0$, or (ib) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) < 0$, $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $A''_{iFF}(\tau_0) - A''_{iFH}(\tau_0) \leq 0$. In Case (ia), the same argument as for τ_0 implies $A''_{iHH}(\tau'_0) - A''_{iHF}(\tau'_0) > 0$, which combined with (B.67), $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFF}(\tau'_0) - A'_{iFH}(\tau'_0) < 0$, a contradiction. In Case (ib), the same argument as for

τ_0 implies $A''_{iFF}(\tau'_0) - A''_{iFH}(\tau'_0) < 0$, which combined with (B.68), $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} > 0$ implies $A'_{iHH}(\tau'_0) - A'_{iHF}(\tau'_0) > 0$, a contradiction. Therefore, τ'_0 is infinite, which means that $A_{iFF}(\tau) - A_{iFH}(\tau)$ is increasing, and $A_{iHH}(\tau) - A_{iHF}(\tau)$ is increasing in $(0, \tau_0)$ and decreasing in (τ_0, ∞).

Consider next Case (ii). A symmetric argument by switching H and F implies that $A_{iHH}(\tau) - A_{iHF}(\tau)$ is increasing, and $A_{iFF}(\tau) - A_{iFH}(\tau)$ is increasing in $(0, \tau_0)$ and decreasing in (τ_0, ∞). ∎

Using Lemmas B.1-B.6, we next prove the proposition. Since (A_{iHe}, A_{iFe}) are positive (Lemma B.5), (3.1) implies $\frac{\partial e_t}{\partial i_{Ht}} < 0$ and $\frac{\partial e_t}{\partial i_{Ft}} > 0$. When $a > 0$ and $\alpha_e > 0$, (B.41) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, and the proof of Lemma B.3 implies that $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are positive. Hence,

$$a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{iFH} A_{iFe} < 0, \tag{B.69}$$

$$a\sigma_{iF}^2 \bar{\lambda}_{iFF} A_{iFe} - a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHe} < 0. \tag{B.70}$$

Equations (B.69) and (B.70) also hold when $a > 0$, $\alpha_e = 0$ and $(\alpha_H(\tau), \alpha_F(\tau))$ are positive. This is because (B.41) again implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, and the proof of Lemma B.3 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$. Combining (B.69) and (B.70) with (B.36), we find $A_{iHe} < \frac{1}{\kappa_{iH}} \equiv A_{iHe}^{UIP}$ and $A_{iFe} < \frac{1}{\kappa_{iF}} \equiv A_{iFe}^{UIP}$. Combining (B.69) and (B.70) with (3.10) (for the definitions of (A_e, λ_t) in Section 4.2) and (B.45), we find $\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ht}} < 0$ and $\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ft}} > 0$. This establishes the first bullet point of the proposition.

Since $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive for all $\tau > 0$ (Lemma B.3), (2.1) and (3.2) imply that $\left(\frac{\partial y_{Ht}^{(\tau)}}{\partial i_{Ht}}, \frac{\partial y_{Ft}^{(\tau)}}{\partial i_{Ft}}\right)$ are positive. When $a > 0$ and $\alpha_e > 0$, Lemma B.3 implies that $(A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$, and Lemma B.4 implies that $(A_{iHF}(\tau), A_{iFH}(\tau))$ are increasing. Equation (B.39) for $(j, j') = (H, F)$ implies

$$a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\tau) + a\sigma_{iF}^2 \bar{\lambda}_{iFF} A_{iFH}(\tau) > 0. \tag{B.71}$$

Multiplying both sides of (B.71) by $\frac{\bar{\lambda}_{iHH}}{\bar{\lambda}_{iHF}} < 0$, we find

$$\begin{aligned} & a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHH}(\tau) + a\sigma_{iF}^2 \frac{\bar{\lambda}_{iHH}\bar{\lambda}_{iFF}}{\bar{\lambda}_{iHF}} A_{iFH}(\tau) < 0 \\ & \Rightarrow a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHH}(\tau) + a\sigma_{iF}^2 \bar{\lambda}_{iFH} A_{iFH}(\tau) < 0, \end{aligned} \tag{B.72}$$

where the second step follows from $A_{iFH}(\tau) > 0$ and from the inequality $\bar{\lambda}_{iHH}\bar{\lambda}_{iFF} - \bar{\lambda}_{iHF}\bar{\lambda}_{iFH} < 0$ established in the proof of Lemma B.1. We likewise find

$$a\sigma_{iF}^2 \bar{\lambda}_{iFH} A_{iFF}(\tau) + a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHF}(\tau) > 0, \tag{B.73}$$

$$\Rightarrow a\sigma_{iF}^2 \bar{\lambda}_{iFF} A_{iFF}(\tau) + a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHF}(\tau) < 0, \tag{B.74}$$

by switching H and F. Equations (B.72) and (B.74) hold also when $a > 0$, $\alpha_e = 0$ and $(\alpha_H(\tau), \alpha_F(\tau))$ are positive. Indeed, the proof of Lemma B.3 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, and since $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive, (B.41) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative. Combining (B.72) and (B.74) with (B.38), we find $A_{iHH}(\tau) < \frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}} \equiv A^{EH}_{iHH}(\tau)$ and $A_{iFF}(\tau) < \frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}} \equiv A^{EH}_{iFF}(\tau)$. Combining (B.72) and (B.74) with (3.11) (for the definitions of $(A_j(\tau), \lambda_t)$ in Section 4.2) and (B.45), we find $\frac{\partial\left(\mu^{(\tau)}_{Ht} - i_{Ht}\right)}{\partial i_{Ht}} < 0$ and $\frac{\partial\left(\mu^{(\tau)}_{Ft} - i_{Ft}\right)}{\partial i_{Ft}} < 0$. This establishes the second bullet point of the proposition.

When $a > 0$ and $\alpha_e > 0$, $(A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$, and hence (2.1) and (3.2) imply that $(\frac{\partial y^{(\tau)}_{Ht}}{\partial i_{Ft}}, \frac{\partial y^{(\tau)}_{Ft}}{\partial i_{Ht}})$ are positive. Moreover, combining (B.71) and (B.73) with (3.11) and (B.45), we find $\frac{\partial\left(\mu^{(\tau)}_{Ht} - i_{Ht}\right)}{\partial i_{Ft}} > 0$ and $\frac{\partial\left(\mu^{(\tau)}_{Ft} - i_{Ft}\right)}{\partial i_{Ht}} > 0$. This establishes the third bullet point of the proposition. The fourth bullet point follows from Lemma B.6, (2.1) and (3.2). ∎

Proof of Proposition 4.5: Combining (3.10) and (3.11) (for the definitions of $(A_e, A_j(\tau), \lambda_t)$ in Section 4.2) with (4.2), we can write the expected return of the hybrid CCT as

$$\mu^{(\tau)}_{hCCTt} \equiv \lambda_{iHt}(A_{iHe} + A_{iFH}(\tau) - A_{iHH}(\tau)) - \lambda_{iFt}(A_{iFe} + A_{iHF}(\tau) - A_{iFF}(\tau)). \tag{B.75}$$

Using (B.45), we find

$$\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} = a\sigma^2_{iH}\bar{\lambda}_{iHH}(A_{iHe} + A_{iHF}(\tau) - A_{iHH}(\tau)) - a\sigma^2_{iF}\bar{\lambda}_{iFH}(A_{iFe} + A_{iFH}(\tau) - A_{iFF}(\tau)), \tag{B.76}$$

$$\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} = a\sigma^2_{iH}\bar{\lambda}_{iHF}(A_{iHe} + A_{iHF}(\tau) - A_{iHH}(\tau)) - a\sigma^2_{iF}\bar{\lambda}_{iFF}(A_{iFe} + A_{iFH}(\tau) - A_{iFF}(\tau)). \tag{B.77}$$

When $a > 0$, and $\alpha_e > 0$ or $\alpha_j(\tau) > 0$, $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative. Since, in addition, $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are non-negative, (A_{iHe}, A_{iFe}) are positive and $A_{iHH}(0) - A_{iHF}(0) = A_{iFF}(0) - A_{iFH}(0) = 0$, (B.76) and (B.77) imply that there exists a threshold $\tau^* > 0$ such that $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} > 0$ for all $\tau \in (0, \tau^*)$. Since at least one of $(A_{iHH}(\tau) - A_{iHF}(\tau), A_{iFF}(\tau) - A_{iFH}(\tau))$ is increasing (proof of Lemma B.4), they are both increasing when countries are symmetric. Since, in addition, $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty)) = (A_{iHe}, A_{iFe})$ (proof of Lemma B.6), (B.76) and (B.77) imply that when countries are symmetric, $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} > 0$ for all $\tau > 0$, which means $\tau^* = \infty$.

Combining

$$\mu_{CCTt} \equiv \mu_{et} + i_{Ft} - i_{Ht} = \lambda_{iHt}A_{iHe} - \lambda_{iFt}A_{iFe},$$

which gives the expected return of the basic CCT and follows from (3.10) (for the definitions of (A_e, λ_t) in Section 4.2), with (B.45), (B.76) and (B.77), we find

$$\frac{\partial\left(\mu^{(\tau)}_{hCCTt} - \mu_{CCTt}\right)}{\partial i_{Ht}} = \bar{\lambda}_{iHH}(A_{iHF}(\tau) - A_{iHH}(\tau)) - \bar{\lambda}_{iHF}(A_{iFH}(\tau) - A_{iFF}(\tau)) > 0, \tag{B.78}$$

$$\frac{\partial\left(\mu^{(\tau)}_{hCCTt} - \mu_{CCTt}\right)}{\partial i_{Ft}} = \bar{\lambda}_{iFH}(A_{iHF}(\tau) - A_{iHH}(\tau)) - \bar{\lambda}_{iFF}(A_{iFH}(\tau) - A_{iFF}(\tau)) < 0, \tag{B.79}$$

where the inequalities follow because $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are non-negative, and $(A_{iHH}(\tau) - A_{iHF}(\tau), A_{iFF}(\tau) - A_{iFH}(\tau))$ are positive for all $\tau > 0$ (Lemma B.6). Hence, the sensitivity of the hybrid CCT's expected return to (i_{Ht}, i_{Ft}) is smaller (less negative in the case of i_{Ht} and less positive in the case of i_{Ft}) than for the basic CCT. Since $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty)) = (A_{iHe}, A_{iFe})$, (B.75) implies that $\mu^{(\tau)}_{hCCTt}$ goes to zero when τ goes to infinity.

Using (3.1), (3.2), (4.3) and $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we can write the return of the long-horizon CCT as

$$\begin{aligned} &A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + C_e + (\pi_F - \pi_H)t - (A_{iHe}i_{H,t+\tau} - A_{iFe}i_{F,t+\tau} + C_e + (\pi_F - \pi_H)(t+\tau)) \\ &+ A_{iFF}(\tau)i_{Ft} + A_{iHF}(\tau)i_{Ht} + C_F(\tau) - (A_{iHH}(\tau)i_{Ht} + A_{iFH}(\tau)i_{Ft} + C_H(\tau)). \end{aligned}$$

Hence, (4.1) implies that the annualized expected return of the long-horizon CCT is

$$\begin{aligned} \mu^{(\tau)}_{\ell CCTt} \equiv \frac{1}{\tau}\Big[&A_{iHe}(1 - e^{-\kappa_{iH}\tau})(i_{Ht} - \bar{i}_H) - A_{iFe}(1 - e^{-\kappa_{iF}\tau})(i_{Ft} - \bar{i}_F) - (\pi_F - \pi_H)\tau \\ &+ A_{iFF}(\tau)i_{Ft} + A_{iHF}(\tau)i_{Ht} + C_F(\tau) - (A_{iHH}(\tau)i_{Ht} + A_{iFH}(\tau)i_{Ft} + C_H(\tau))\Big], \end{aligned} \tag{B.80}$$

and its sensitivity to (i_{Ht}, i_{Ft}) is

$$\frac{\partial\mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} = \frac{1}{\tau}\left[A_{iHe}(1 - e^{-\kappa_{iH}\tau}) + A_{iHF}(\tau) - A_{iHH}(\tau)\right], \tag{B.81}$$

$$\frac{\partial\mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} = \frac{1}{\tau}\left[-A_{iFe}(1 - e^{-\kappa_{iF}\tau}) + A_{iFF}(\tau) - A_{iFe}(\tau)\right]. \tag{B.82}$$

When $a > 0$, and $\alpha_e > 0$ or $\alpha_j(\tau) > 0$, $A_{iHe} < \frac{1}{\kappa_{iH}}$ and $A_{iFe} < \frac{1}{\kappa_{iF}}$. Since, in addition, $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$, the derivative of (B.81) with respect to τ at $\tau = 0$ is negative, and the derivative of (B.82) with respect to τ at $\tau = 0$ is positive. Hence, there exists a threshold $\tau^* > 0$ such that $\frac{\partial\mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial\mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} > 0$ for all $\tau \in (0, \tau^*)$. When countries are symmetric, we set $\kappa_r \equiv \kappa_{iH} = \kappa_{iF}$, $\sigma_r \equiv \sigma_{iH} = \sigma_{iF}$, $A_{ie} \equiv A_{iHe} = A_{iFe}$, $\Delta A(\tau) \equiv A_{iHH}(\tau) - A_{iHF}(\tau) = A_{iFF}(\tau) - A_{iFH}(\tau)$,

$\Delta\bar{\lambda} \equiv \bar{\lambda}_{iHH} - \bar{\lambda}_{iHF} = \bar{\lambda}_{iFF} - \bar{\lambda}_{iFH} < 0$. Taking the difference between (B.38) and (B.39) yields

$$\Delta A'(\tau) + \kappa_r \Delta A(\tau) - 1 = a\sigma_r^2 \Delta\bar{\lambda}\Delta A(\tau),$$

which integrates to

$$\Delta A(\tau) = A_{ie}\left(1 - e^{-(\kappa_r - a\sigma_r^2\Delta\bar{\lambda})\tau}\right)$$

since $\Delta A(0) = 0$ and $\Delta A(\infty) = A_{ie}$. Substituting into (B.81) and (B.82), we find

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} = -\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} = \frac{1}{\tau} A_{ie}(e^{-(\kappa_r - a\sigma_r^2\Delta\bar{\lambda})\tau} - e^{-\kappa_r\tau}) < 0. \tag{B.83}$$

Hence, $\tau^* = \infty$.

The annualized expected return of the sequence of basic CCTs is

$$\mu^{(\tau)}_{CCTt} \equiv \frac{1}{\tau} E_t \int_t^{t+\tau} (\lambda_{iHt'} A_{iHe} - \lambda_{iFt'} A_{iFe})\, dt'.$$

Using (4.1) and (B.45), we find

$$\begin{aligned}\frac{\partial \mu^{(\tau)}_{CCTt}}{\partial i_{Ht}} &= \frac{1 - e^{-\kappa_{iH}\tau}}{\kappa_{iH}\tau}\left(a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{iFH} A_{iFe}\right) \\ &= \frac{1 - e^{-\kappa_{iH}\tau}}{\kappa_{iH}\tau}(\kappa_{iH} A_{iHe} - 1),\end{aligned} \tag{B.84}$$

where the second step follows from (B.36). We likewise find

$$\frac{\partial \mu^{(\tau)}_{CCTt}}{\partial i_{Ft}} = -\frac{1 - e^{-\kappa_{iF}\tau}}{\kappa_{iF}\tau}(\kappa_{iF} A_{iFe} - 1). \tag{B.85}$$

Combining (B.81) and (B.84), we find

$$\frac{\partial\left(\mu^{(\tau)}_{\ell CCTt} - \mu^{(\tau)}_{CCTt}\right)}{\partial i_{Ht}} = \frac{1}{\tau}\left[\frac{1 - e^{-\kappa_{iH}\tau}}{\kappa_{iH}} + A_{iHF}(\tau) - A_{iHH}(\tau)\right] > 0,$$

where the inequality sign follows from (B.65) by noting that the left-hand side of (B.65) is negative. Combining (B.82) and (B.85), we likewise find

$$\frac{\partial\left(\mu^{(\tau)}_{\ell CCTt} - \mu^{(\tau)}_{CCTt}\right)}{\partial i_{Ft}} = \frac{1}{\tau}\left[-\frac{1 - e^{-\kappa_{iF}\tau}}{\kappa_{iF}} + A_{iFF}(\tau) - A_{iFH}(\tau)\right] < 0.$$

Hence, the sensitivity of the long-horizon CCT's expected return to (i_{Ht}, i_{Ft}) is smaller (less negative in the case of i_{Ht} and less positive in the case of i_{Ft}) than for the corresponding sequence of basic CCTs. Since $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ go to finite limits when τ goes to infinity, (B.80) implies that $\mu^{(\tau)}_{\ell CCTt}$ goes to

$$
\begin{aligned}
&\lim_{\tau\to\infty}\frac{C_F(\tau)}{\tau}-\lim_{\tau\to\infty}\frac{C_H(\tau)}{\tau}-(\pi_F-\pi_H)\\
&=\kappa_{iF}\bar{i}_F\left(A_{iFF}(\infty)-A_{iFH}(\infty)\right)-\kappa_{iH}\bar{i}_H\left(A_{iHH}(\infty)-A_{iHF}(\infty)\right)\\
&\quad-\frac{1}{2}\sigma^2_{iF}\left[A_{iFF}(\infty)\left(A_{iFF}(\infty)-2A_{iFe}\right)-A_{iFH}(\infty)^2\right]\\
&\quad+\frac{1}{2}\sigma^2_{iH}\left[A_{iHH}(\infty)^2-A_{iHF}(\infty)\left(A_{iHF}(\infty)+2A_{iHe}\right)\right]\\
&\quad+a\sigma^2_{iF}\bar{\lambda}_{iFC}\left(A_{iFF}(\infty)-A_{iFH}(\infty)\right)-a\sigma^2_{iH}\bar{\lambda}_{iHC}\left(A_{iHH}(\infty)-A_{iHF}(\infty)\right)-(\pi_F-\pi_H)\\
&=\kappa_{iF}\bar{i}_F A_{iFe}-\kappa_{iH}\bar{i}_H A_{iHe}+\frac{1}{2}\sigma^2_{iF}A^2_{iFe}+\frac{1}{2}\sigma^2_{iH}A^2_{iHe}\\
&\quad+a\sigma^2_{iF}\bar{\lambda}_{iFC}A_{iFe}-a\sigma^2_{iH}\bar{\lambda}_{iHC}A_{iHe}-(\pi_F-\pi_H)=0,
\end{aligned}
$$

where the second step follows from (B.40) by noting $\lim_{\tau\to\infty}\frac{C_j(\tau)}{\tau}=\lim_{\tau\to\infty}C'_j(\tau)$, the third step follows from $(A_{iHH}(\infty)-A_{iHF}(\infty), A_{iFF}(\infty)-A_{iFH}(\infty))=(A_{iHe}, A_{iFe})$, and the fourth step follows from (B.37). Since

$$
\lim_{\tau\to\infty}\frac{C_F(\tau)}{\tau}-\lim_{\tau\to\infty}\frac{C_H(\tau)}{\tau}-(\pi_F-\pi_H)=y^{(\infty)}_F-y^{(\infty)}_H-(\pi_F-\pi_H),
$$

the difference in real yields across countries becomes zero in the limit τ goes to infinity. ∎

We next prove a lemma that we use in subsequent proofs.

Lemma B.7. *When $a>0$ and $\alpha_e>0$, the functions $\left(\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}, \frac{A_{iHF}(\tau)}{A_{iFF}(\tau)}\right)$ are increasing.*

Proof: The functions $(A_{iHH}(\tau), A_{iFH}(\tau))$ solve the system (B.51) of linear ODEs with constant coefficients. The solution is an affine function of $(e^{-\nu_1\tau}, e^{-\nu_2\tau})$, where (ν_1,ν_2) are the eigenvalues of the matrix M. Because of the initial conditions $A_{iHH}(0)=A_{iFH}(0)=0$, we can write the solution as a linear function of $\left(\frac{1-e^{-\nu_1\tau}}{\nu_1}, \frac{1-e^{-\nu_2\tau}}{\nu_2}\right)$. Because $(A'_{iHH}(0), A'_{iFH}(0))=(1,0)$, the coefficients of the linear terms sum to one for $A_{iHH}(\tau)$ and to zero for $A_{iFH}(\tau)$. Hence, we can write the solution as

$$
A_{iHH}(\tau)=\frac{1-e^{-\nu_1\tau}}{\nu_1}+\phi_{HH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2}-\frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{B.86}
$$

$$
A_{iFH}(\tau)=\phi_{FH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2}-\frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{B.87}
$$

for scalars (ϕ_{HH},ϕ_{FH}). The eigenvalues (ν_1,ν_2) are positive (Lemma B.1), and without loss of generality we can set $\nu_1>\nu_2$. Since $A_{iFH}(\tau)$ is positive when $a>0$ and $\alpha_e>0$

(Lemma B.3), $\phi_{FH} > 0$. Since

$$\frac{A_{iHH}(\tau)}{A_{iFH}(\tau)} = \frac{\frac{1-e^{-\nu_1\tau}}{\nu_1}}{\phi_{FH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right)} + \frac{\phi_{HH}}{\phi_{FH}} = \frac{1}{\phi_{HF}\left(\frac{\nu_1}{\nu_2}\frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}} - 1\right)} + \frac{\phi_{HH}}{\phi_{FH}},$$

and the function $(\nu_1, \nu_2, \tau) \longrightarrow \frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}}$ increases in τ because its derivative has the same sign as $\frac{e^{\nu_1\tau}-1}{\nu_1} - \frac{e^{\nu_2\tau}-1}{\nu_2}$, the function $\frac{A_{iHH}(\tau)}{A_{FH}(\tau)}$ is decreasing. Hence, the inverse function $\frac{A_{iFH}(\tau)}{A_{HH}(\tau)}$ is increasing. A similar argument using (B.52) establishes that $\frac{A_{iHF}(\tau)}{A_{FF}(\tau)}$ is increasing. ∎

Proof of Proposition 4.6: Consider a one-off increase in γ_t at time zero, and denote by κ_γ the rate at which γ_t reverts to its mean of zero. Equation (B.45) is modified to

$$\begin{aligned}
\lambda_{ijt} = a\sigma_{ij}^2 \Bigg(& \left[\zeta_e + \theta_e\gamma_t - \alpha_e\left(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}\gamma_t + C_e\right)\right] A_{ije}(-1)^{1_{\{j=F\}}} \\
& + \sum_{j'=H,F}\int_0^T \left[\zeta_{j'}(\tau) - \alpha_{j'}(\tau)\left(A_{iHj'}(\tau)i_{Ht} + A_{iFj'}(\tau)i_{Ft} + A_{\gamma j'}(\tau)\gamma_t + C_{j'}(\tau)\right)\right] A_{ijj'}(\tau)d\tau \Bigg) \\
\equiv {} & a\sigma_{ij}^2\left(\bar{\lambda}_{ijj}i_{jt} + \bar{\lambda}_{ijj'}i_{j't} + \bar{\lambda}_{ij\gamma}\gamma_t + \bar{\lambda}_{ijC}\right)
\end{aligned} \tag{B.88}$$

(B.12) is modified to (B.28), and (3.7) and (3.8) are modified to

$$\begin{aligned}
\mu_{jt}^{(\tau)} \equiv & A'_{iHj}(\tau)i_{Ht} + A'_{iFj}(\tau)i_{Ft} + A'_{\gamma j}(\tau)\gamma_t + C'_j(\tau) \\
& - A_{iHj}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) - A_{iFj}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\gamma j}(\tau)\kappa_\gamma\gamma_t \\
& + \frac{1}{2}A_{iHj}(\tau)\left(A_{iHj}(\tau) + 2A_{iHe}1_{\{j=F\}}\right)\sigma_{iH}^2 + \frac{1}{2}A_{iFj}(\tau)\left(A_{iFj}(\tau) - 1_{\{j=F\}}2A_{iFe}\right)\sigma_{iF}^2.
\end{aligned} \tag{B.89}$$

Substituting λ_t from (B.88) and μ_{et} from (B.28) into (3.10) (for the definitions of (A_e, λ_t) in Section 4.2), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$\kappa_\gamma A_{\gamma e} = a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}A_{iHe} - a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}A_{iFe}. \tag{B.90}$$

Substituting λ_t from (B.88) and $\mu_{jt}^{(\tau)}$ from (B.89) into (3.11) (for the definitions of $(A_j(\tau), \lambda_t)$ in Section 4.2), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$A'_{\gamma j}(\tau) + \kappa_\gamma A_{\gamma j}(\tau) = a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}A_{iHj}(\tau) + a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}A_{iFj}(\tau). \tag{B.91}$$

Solving (B.91) with the initial condition $A_{\gamma j}(0)=0$, we find

$$A_{\gamma j}(\tau)=a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}\int_0^\tau A_{iHj}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'+a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}\int_0^\tau A_{iFj}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau', \quad \text{(B.92)}$$

We next substitute $A_{\gamma e}$ from (B.90) and $\{A_{\gamma j}(\tau)\}_{j=H,F}$ from (B.92) into

$$\bar{\lambda}_{ij\gamma}\equiv(\theta_e-\alpha_e A_{\gamma e})A_{ije}(-1)^{1_{\{j=F\}}}-\int_0^T\alpha_H(\tau)A_{\gamma H}(\tau)A_{ijH}(\tau)d\tau-\int_0^T\alpha_F(\tau)A_{\gamma F}(\tau)A_{ijF}(\tau)d\tau, \quad \text{(B.93)}$$

which follows from the definition of $\bar{\lambda}_{ij\gamma}$ in (B.88). We find

$$(1+a\sigma_{iH}^2 z_{\gamma HH})\bar{\lambda}_{iH\gamma}+a\sigma_{iF}^2 z_{\gamma FH}\bar{\lambda}_{iF\gamma}=\theta_e A_{iHe}, \quad \text{(B.94)}$$
$$a\sigma_{iH}^2 z_{\gamma HF}\bar{\lambda}_{iH\gamma}+(1+a\sigma_{iF}^2 z_{\gamma FF})\bar{\lambda}_{iF\gamma}=-\theta_e A_{iFe}, \quad \text{(B.95)}$$

where

$$\begin{aligned}
z_{\gamma HH}&=\frac{\alpha_e}{\kappa_\gamma}A_{iHe}^2+\int_0^T\alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau\\
&\quad+\int_0^T\alpha_F(\tau)A_{iHF}(\tau)\left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau,\\
z_{\gamma FF}&=\frac{\alpha_e}{\kappa_\gamma}A_{iFe}^2+\int_0^T\alpha_H(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau\\
&\quad+\int_0^T\alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau,
\end{aligned}$$

$$\begin{aligned}
z_{\gamma HF}&=-\frac{\alpha_e}{\kappa_\gamma}A_{iHe}A_{iFe}+\int_0^T\alpha_H(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau\\
&\quad+\int_0^T\alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau,\\
z_{\gamma FH}&=-\frac{\alpha_e}{\kappa_\gamma}A_{iHe}A_{iFe}+\int_0^T\alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau\\
&\quad+\int_0^T\alpha_F(\tau)A_{iHF}(\tau)\left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau.
\end{aligned}$$

Equations (B.94) and (B.95) form a linear system of two equations in the two unknowns $(\bar{\lambda}_{iH\gamma},\bar{\lambda}_{iF\gamma})$. Its solution is

$$\bar{\lambda}_{iH\gamma}=\frac{\theta_e}{\Delta_{z\gamma}}\left[(1+a\sigma_{iF}^2 z_{\gamma FF})A_{iHe}+a\sigma_{iF}^2 z_{\gamma FH}A_{iFe}\right] \quad \text{(B.96)}$$
$$\bar{\lambda}_{iF\gamma}=-\frac{\theta_e}{\Delta_{z\gamma}}\left[(1+a\sigma_{iH}^2 z_{\gamma HH})A_{iFe}+a\sigma_{iH}^2 z_{\gamma HF}A_{iHe}\right], \quad \text{(B.97)}$$

where

$$\Delta_{z\gamma} \equiv (1 + a\sigma_{iH}^2 z_{\gamma HH})(1 + a\sigma_{iF}^2 z_{\gamma FF}) - a^2\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma HF} z_{\gamma FH}.$$

To complete the proof, we proceed in three steps. In Step 1, we show that $\Delta_{z\gamma}$ is positive. In Step 2, we show that $A_{\gamma e}$ is positive. This proves the first statement in the proposition. In Step 3, we show that $A_{\gamma H}(\tau)$ is positive and $A_{\gamma F}(\tau)$ is negative. This proves the second and third statements in the proposition.

Step 1: $\Delta_{z\gamma}$ is positive. Since $(z_{\gamma HH}, z_{\gamma FF})$ are non-negative, $\Delta_{z\gamma} > 0$ under the sufficient condition

$$z_{\gamma HH} z_{\gamma FF} \geq z_{\gamma HF} z_{\gamma FH}. \tag{B.98}$$

The function

$$\begin{aligned} F(\mu) &\equiv z_{\gamma HH} + \mu(z_{\gamma HF} + z_{\gamma FH}) + \mu^2 z_{\gamma FF} \\ &= \frac{\alpha_e}{\kappa_\gamma}\left(A_{iHe} - \mu A_{iFe}\right)^2 \\ &\quad + \int_0^T \alpha_H(\tau)\left[A_{iHH}(\tau) + \mu A_{iFH}(\tau)\right]\left[\int_0^T \left[A_{iHH}(\tau) + \mu A_{iFH}(\tau)\right] e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\ &\quad + \int_0^T \alpha_F(\tau)\left[A_{iHF}(\tau) + \mu A_{iFF}(\tau)\right]\left[\int_0^T \left[A_{iHF}(\tau) + \mu A_{iFF}(\tau)\right] e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \end{aligned}$$

is non-negative for all μ if

$$F_0 \equiv \int_0^T \alpha(\tau)A(\tau)\left[\int_0^\tau A(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau$$

is non-negative for a non-negative and non-increasing $\alpha(\tau)$. Since

$$F_0 = \int_0^T \phi(\tau)\Phi(\tau)\left[\int_0^\tau \Phi(\tau')d\tau'\right] d\tau,$$

where

$$\begin{aligned} \phi(\tau) &\equiv \alpha(\tau)e^{-2\kappa_\gamma \tau}, \\ \Phi(\tau) &\equiv A(\tau)e^{\kappa_\gamma \tau}, \end{aligned}$$

integration by parts implies

$$F_0 = \frac{1}{2}\phi(T)\left[\int_0^T \Phi(\tau)d\tau\right]^2 - \frac{1}{2}\int_0^T \phi'(\tau)\left[\int_0^\tau \Phi(\tau')d\tau'\right]^2 d\tau. \tag{B.99}$$

The first term in the right-hand side of (B.99) is non-negative because $\alpha(\tau)$ is non-negative, and the first term is non-positive because $\alpha(\tau)$ is non-increasing. Therefore, F_0 is non-negative. Since $F(\mu)$ is quadratic in μ, its non-negativity for all μ implies

$$
\begin{aligned}
&4z_{\gamma HH}z_{\gamma FF} \geq (z_{\gamma HF}+z_{\gamma FH})^2 \\
&\Rightarrow z_{\gamma HH}z_{\gamma FF} \geq \frac{1}{4}(z_{\gamma HF}+z_{\gamma FH})^2 = z_{\gamma HF}z_{\gamma FH} + \frac{1}{4}(z_{\gamma HF}-z_{\gamma FH})^2 \geq z_{\gamma HF}z_{\gamma FH}.
\end{aligned}
$$

Therefore, (B.98) holds.

Step 2: $A_{\gamma e}(\tau)$ is positive. Substituting $(\bar{\lambda}_{iH\gamma}, \bar{\lambda}_{iF\gamma})$ from (B.96) and (B.97) into (B.90), and using the definitions of $(z_{\gamma HH}, z_{\gamma FF}, z_{\gamma HF}, z_{\gamma FH})$ and that $(\theta_e, \Delta_{z\gamma})$ are positive, we find $A_{\gamma e} > 0$ if

$$
Z_{\gamma H}A_{iHe} + Z_{\gamma F}A_{iFe} > 0, \tag{B.100}
$$

where

$$
\begin{aligned}
Z_{\gamma H} &\equiv \sigma_{iH}^2(1+a\sigma_{iF}^2 z_{\gamma FF})A_{iHe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma FH}A_{iFe} \\
&= \sigma_{iH}^2 A_{iHe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2\int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau)+A_{iFe}A_{iHH}(\tau)]\left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2\int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau)+A_{iFe}A_{iHF}(\tau)]\left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau, \\
Z_{\gamma F} &\equiv \sigma_{iF}^2(1+a\sigma_{iH}^2 z_{\gamma HH})A_{iFe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma HF}A_{iHe} \\
&= \sigma_{iF}^2 A_{iFe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2\int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau)+A_{iFe}A_{iHH}(\tau)]\left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2\int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau)+A_{iFe}A_{iHF}(\tau)]\left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau.
\end{aligned}
$$

Since $(A_{iHe}, A_{iFe}, Z_{\gamma H}, Z_{\gamma F})$ are positive, (B.100) holds.

Step 3: $A_{\gamma H}(\tau)$ is positive and $A_{\gamma F}(\tau)$ is negative. We prove that $A_{\gamma H}(\tau)$ is positive. The proof that $A_{\gamma F}(\tau)$ is negative is symmetric. Substituting $(\bar{\lambda}_{iH\gamma}, \bar{\lambda}_{iF\gamma})$ from (B.96) and (B.97) into (B.92) for $j = H$, and using the definitions of $(z_{\gamma HH}, z_{\gamma FF}, z_{\gamma HF}, z_{\gamma FH})$ and that $(\theta_e, \Delta_{z\gamma})$ are positive, we find $A_{\gamma H}(\tau) > 0$ if

$$
Z_{\gamma H}\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' - Z_{\gamma F}\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' > 0. \tag{B.101}
$$

Since $(A_{iHH}(\tau), Z_{\gamma H}, Z_{\gamma F})$ are positive, $A_{iFH}(\tau)$ is non-negative and $\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing, (B.101) holds under the sufficient condition

$$
Z_{\gamma H}A_{iHH}(\infty) - Z_{\gamma F}A_{iFH}(\infty) > 0. \tag{B.102}
$$

Using the definitions of $(Z_{\gamma H}, Z_{\gamma F})$, we can write (B.102) as

$$
\begin{aligned}
&\sigma_{iH}^2 A_{iHe} A_{iHH}(\infty) - \sigma_{iF}^2 A_{iFe} A_{iFH}(\infty) \\
&+ a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe} A_{iFH}(\tau) + A_{iFe} A_{iHH}(\tau)] \\
&\times \left[\int_0^\tau \left[A_{iFH}(\tau') A_{iHH}(\infty) - A_{iHH}(\tau') A_{iFH}(\infty)\right] e^{-\kappa_\gamma(\tau - \tau')} d\tau' \right] d\tau \\
&+ a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe} A_{iFF}(\tau) + A_{iFe} A_{iHF}(\tau)] \\
&\times \left[\int_0^\tau \left[A_{iFF}(\tau') A_{iHH}(\infty) - A_{iHF}(\tau') A_{iFH}(\infty)\right] e^{-\kappa_\gamma(\tau - \tau')} d\tau' \right] d\tau > 0.
\end{aligned} \tag{B.103}
$$

Equation (B.39) for $(j, j') = (H, F)$ implies

$$
A_{iFH}(\tau) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}} - \frac{A'_{iFH}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}}, \tag{B.104}
$$

which for $\tau = \infty$ becomes

$$
A_{iFH}(\infty) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\infty)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}}. \tag{B.105}
$$

Equation (B.38) for $(j, j') = (F, H)$ implies

$$
A_{iFF}(\tau) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHF}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}} + \frac{1 - A'_{iFF}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}}. \tag{B.106}
$$

Using (B.104)-(B.106) to simplify the terms in the first, second and fourth lines of (B.103), and dividing throughout by $\frac{a\sigma_{iH}^2 \sigma_{iF}^2 A_{iHH}(\infty)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}} > 0$, we find that (B.103) is equivalent to

$$
\begin{aligned}
&\left(\frac{\kappa_{iF}}{a\sigma_{iF}^2} - \bar{\lambda}_{iFF} \right) A_{iHe} - \bar{\lambda}_{iHF} A_{iFe} \\
&- \int_0^T \alpha_H(\tau)[A_{iHe} A_{iFH}(\tau) + A_{iFe} A_{iHH}(\tau)] \left[\int_0^\tau A'_{iFH}(\tau') e^{-\kappa_\gamma(\tau - \tau')} d\tau' \right] d\tau \\
&+ \int_0^T \alpha_F(\tau)[A_{iHe} A_{iFF}(\tau) + A_{iFe} A_{iHF}(\tau)] \left[\int_0^\tau (1 - A'_{iFF}(\tau')) e^{-\kappa_\gamma(\tau - \tau')} d\tau' \right] d\tau > 0.
\end{aligned} \tag{B.107}
$$

Equations (B.41) and (B.42) imply

$$
\begin{aligned}
&- \bar{\lambda}_{iFF} A_{iHe} - \bar{\lambda}_{iHF} A_{iFe} \\
&= \int_0^T \alpha_H(\tau) A_{iFH}(\tau)[A_{iHe} A_{iFH}(\tau) + A_{iFe} A_{iHH}(\tau)] d\tau
\end{aligned}
$$

$$+\int_0^T \alpha_F(\tau)A_{iFF}(\tau)[A_{iHe}A_{iFF}(\tau)+A_{iFe}A_{iHF}(\tau)]d\tau. \tag{B.108}$$

We next substitute (B.108) into (B.107). Noting that $1-A'_{iFF}(\tau)>0$, which follows from (B.38) for $(j,j')=(F,H)$ and (B.74), and that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHe}, A_{iFe})$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$ are non-negative, we find that (B.107) holds under the sufficient condition

$$\int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau)+A_{iFe}A_{iHH}(\tau)]\left[A_{FH}(\tau)-\int_0^\tau A'_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau \geq 0,$$

which, in turn, holds because

$$A_{iFH}(\tau)-\int_0^\tau A'_{iFH}(\tau)e^{-\kappa_\gamma(\tau-\tau')}d\tau' \geq A_{iFH}(\tau)-\int_0^\tau A'_{iFH}(\tau')d\tau' = A_{iFH}(0)=0.$$

∎

Proof of Proposition 4.7: We prove the proposition in the case $j=H$. The proof for the case $j=F$ is symmetric. Consider a one-off increase in β_{Ht} at time zero, and denote by $\kappa_{\beta H}$ the rate at which β_{Ht} reverts to its mean of zero. The counterparts of (B.90) and (B.92) are

$$\kappa_{\beta H}A_{\beta He} = a\sigma_{iH}^2\bar{\lambda}_{iH\beta}A_{iHe} - a\sigma_{iF}^2\bar{\lambda}_{iF\beta}A_{iFe}, \tag{B.109}$$

$$A_{\beta Hj}(\tau) = a\sigma_{iH}^2\bar{\lambda}_{iH\beta}\int_0^\tau A_{iHj}(\tau')e^{-\kappa_{\beta H}(\tau-\tau')}d\tau' + a\sigma_{iF}^2\bar{\lambda}_{iF\beta}\int_0^\tau A_{iFj}(\tau')e^{-\kappa_{\beta H}(\tau-\tau')}d\tau, \tag{B.110}$$

where

$$\bar{\lambda}_{ij\beta} \equiv -\alpha_e A_{\gamma e}A_{ije}(-1)^{1_{\{j=F\}}} + \int_0^T [\theta_H(\tau)-\alpha_H(\tau)A_{\beta HH}(\tau)]\,A_{ijH}(\tau)d\tau - \int_0^T \alpha_F(\tau)A_{\beta HF}(\tau)A_{ijF}(\tau)d\tau \tag{B.111}$$

is the counterpart of (B.93). The counterparts of (B.94) and (B.95) are

$$(1+a\sigma_{iH}^2 z_{\beta HH})\bar{\lambda}_{iH\beta} + a\sigma_{iF}^2 z_{\beta FH}\bar{\lambda}_{iF\beta} = \int_0^T \theta_H(\tau)A_{iHH}(\tau)d\tau, \tag{B.112}$$

$$a\sigma_{iH}^2 z_{\beta HF}\bar{\lambda}_{iH\beta} + (1+a\sigma_{iF}^2 z_{\beta FF})\bar{\lambda}_{iF\beta} = \int_0^T \theta_H(\tau)A_{iFH}(\tau)d\tau, \tag{B.113}$$

respectively, where

$$
\begin{aligned}
z_{\beta HH} &= \frac{\alpha_e}{\kappa_{\beta H}} A_{iHe}^2 + \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau \\
&+ \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau, \\
z_{\beta FF} &= \frac{\alpha_e}{\kappa_{\beta H}} A_{iFe}^2 + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau \\
&+ \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau,
\end{aligned}
$$

$$
\begin{aligned}
z_{\beta HF} &= -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau \\
&+ \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau, \\
z_{\beta FH} &= -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau \\
&+ \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau.
\end{aligned}
$$

The solution to the linear system of (B.94) and (B.95) is

$$
\bar{\lambda}_{iH\beta} = \frac{1}{\Delta_{z\beta}} \left[(1 + a\sigma_{iF}^2 z_{\beta FF}) \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau - a\sigma_{iF}^2 z_{\beta FH} \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau \right], \tag{B.114}
$$

$$
\bar{\lambda}_{iF\beta} = \frac{1}{\Delta_{z\beta}} \left[(1 + a\sigma_{iH}^2 z_{\beta HH}) \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau - a\sigma_{iH}^2 z_{\beta HF} \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau \right], \tag{B.115}
$$

where

$$
\Delta_{z\beta} \equiv (1 + a\sigma_{iH}^2 z_{\beta HH})(1 + a\sigma_{iF}^2 z_{\beta FF}) - a^2 \sigma_{iH}^2 \sigma_{iF}^2 z_{\beta HF} z_{\beta FH}.
$$

The same argument as in the proof of Proposition 4.6 implies $\Delta_{z\beta} > 0$.

To complete the proof, we proceed in three steps. In Step 1, we show that $(z_{\beta HF}, z_{\beta FH})$ are non-positive, and are zero when $\alpha_e = 0$. In Step 2, we show that $A_{\beta HH}(\tau)$ is positive, and that $A_{\beta HF}(\tau)$ is positive when $\alpha_e > 0$ and zero when $\alpha_e = 0$. This proves the first and second statements in the proposition. In Step 3, we show that $A_{\beta He}$ is positive. This proves the third statement in the proposition.

Step 1: $(z_{\beta HF}, z_{\beta FH})$ **are non-positive, and are zero when** $\alpha_e = 0$**.** Since Lemma B.3 implies that $A_{iFH}(\tau)$ is non-negative and $A_{iFF}(\tau)$ is positive, and Lemma B.4 implies that $A_{iHH}(\tau)$ is increasing and $A_{iHF}(\tau)$ is non-decreasing,

$$
\begin{aligned}
z_{\beta HF} &\le -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe}A_{iFe} + \int_0^T \alpha_H(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iHH}(\tau)e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
&\quad + \int_0^T \alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau A_{iHF}(\tau)e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
&\le -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe}A_{iFe} + \int_0^T \alpha_H(\tau)A_{iFH}(\tau)\frac{A_{iHH}(\tau)}{\kappa_{\beta H}}d\tau + \int_0^T \alpha_F(\tau)A_{iFF}(\tau)\frac{A_{iHF}(\tau)}{\kappa_{\beta H}} \\
&= -\frac{\bar\lambda_{iHF}}{\kappa_{\beta H}} \le 0,
\end{aligned}
$$

where the second step follows because $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$ are non-negative, the third step follows from (B.42), and the fourth step follows from Lemma B.2. The inequality $z_{\beta FH} \le 0$ follows similarly.

When $\alpha_e = 0$, Lemma B.3 implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Therefore, $z_{\beta HF} = z_{\beta FH} = 0$.

Step 2: $A_{\beta HH}(\tau)$ is positive, and $A_{\beta HF}(\tau)$ is positive when $\alpha_e > 0$ and zero when $\alpha_e = 0$. Since $(\Delta_{z\beta}, \theta_H(\tau), A_{iHH}(\tau))$ are positive, $(A_{iFH}(\tau), z_{\beta FF})$ are non-negative, and $z_{\beta FH} \le 0$, (B.114) implies $\bar\lambda_{iH\beta} > 0$. When $\alpha_e > 0$, $A_{iFH}(\tau) > 0$. Since, in addition, $z_{\beta HH} \ge 0$ and $z_{\beta FH} \le 0$, (B.115) implies $\bar\lambda_{iF\beta} > 0$. When $\alpha_e = 0$, (B.115) and $A_{iFH}(\tau) = z_{\beta HF} = 0$ imply $\bar\lambda_{iF\beta} = 0$.

Since $(\bar\lambda_{iH\beta}, A_{iHH}(\tau))$ are positive and $(\bar\lambda_{iF\beta}, A_{iFH}(\tau))$ are non-negative, (B.110) implies $A_{\beta HH}(\tau) > 0$. When $\alpha_e > 0$, $A_{iHF}(\tau) > 0$. Since, in addition, $(\bar\lambda_{iH\beta}, \bar\lambda_{iF\beta}, A_{iFF}(\tau))$ are positive, (B.110) implies $A_{\beta HF}(\tau) > 0$. When $\alpha_e = 0$, (B.110) and $A_{iHF}(\tau) = \bar\lambda_{iF\beta} = 0$ imply $A_{\beta HF}(\tau) = 0$.

Step 3: $A_{\beta He}$ is positive. Substituting $(\bar\lambda_{\beta HH}, \bar\lambda_{\beta HF})$ from (B.114) and (B.115) into (B.109), and using the definitions of $(z_{\beta HH}, z_{\beta HF}, z_{\beta FH}, z_{\beta FF})$, we find $A_{\beta He} > 0$ if

$$
Z_{\beta H}\int_0^T \theta_H(\tau)A_{iHH}(\tau)d\tau - Z_{\beta F}\int_0^T \theta_H(\tau)A_{iFH}(\tau)d\tau > 0, \tag{B.116}
$$

where

$$
\begin{aligned}
Z_{\beta H} &\equiv \sigma_{iH}^2(1 + a\sigma_{iF}^2 z_{\beta FF})A_{iHe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{\beta HF}A_{iFe} \\
&= \sigma_{iH}^2 A_{iHe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)A_{iFH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau, \\
Z_{\beta F} &\equiv \sigma_{iF}^2(1 + a\sigma_{iH}^2 z_{HH})A_{iFe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{HF}A_{iHe} \\
&= \sigma_{iF}^2 A_{iFe}
\end{aligned}
$$

$$+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right] d\tau$$
$$+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)A_{iHF}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right] d\tau.$$

Since $(\theta_H(\tau), A_{iHH}(\tau))$ are positive, $A_{iFH}(\tau)$ is non-negative, and $\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing (increasing when $a > 0$ and $\alpha_e > 0$ from Lemma B.7, and zero when $a = 0$ or $\alpha_e = 0$), the ratio $\frac{\int_0^T \theta_H(\tau)A_{iFH}(\tau)d\tau}{\int_0^T \theta_H(\tau)A_{iHH}(\tau)d\tau}$ is bounded above by $\frac{A_{iFH}(\infty)}{A_{iHH}(\infty)}$. Since, in addition $(Z_{\beta H}, Z_{\beta F})$ are positive, (B.116) holds for all positive functions $\theta_H(\tau)$ under the sufficient condition

$$Z_{\beta H}A_{iHH}(\infty) - Z_{\beta F}A_{iFH}(\infty) > 0. \tag{B.117}$$

Using the definitions of $(Z_{\beta H}, Z_{\beta F})$, we can write (B.117) as

$$\begin{aligned}
&\sigma_{iH}^2 A_{iHe}A_{iHH}(\infty) - \sigma_{iF}^2 A_{iFe}A_{iFH}(\infty)\\
&+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)\left[A_{iFH}(\tau)A_{iHH}(\infty) - A_{iHH}(\tau)A_{iFH}(\infty)\right]\\
&\times \left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right] d\tau\\
&+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)\left[A_{iFF}(\tau)A_{iHH}(\infty) - A_{iHF}(\tau)A_{iFH}(\infty)\right]\\
&\times \left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right] d\tau > 0.
\end{aligned} \tag{B.118}$$

Using (B.104)-(B.106) to simplify the terms in the first, second and fourth lines of (B.118), and dividing throughout by $\frac{a\sigma_{iH}^2\sigma_{iF}^2 A_{iHH}(\infty)}{\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}} > 0$, we find that (B.118) is equivalent to

$$\begin{aligned}
&\left(\frac{\kappa_{iF}}{a\sigma_{iF}^2} - \bar{\lambda}_{iFF}\right)A_{iHe} - \bar{\lambda}_{iHF}A_{iFe}\\
&- \int_0^T \alpha_H(\tau)A'_{iFH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right] d\tau\\
&+ \int_0^T \alpha_F(\tau)(1 - A'_{iFF}(\tau))\left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right] d\tau > 0.
\end{aligned} \tag{B.119}$$

We next substitute (B.108) into (B.119). Noting that $1 - A'_{iFF}(\tau) > 0$ and that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHe}, A_{iFe})$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$ are non-negative, we find that (B.119) holds under the sufficient condition

$$\begin{aligned}
&\int_0^T \alpha_H(\tau)\Big\{A_{iFH}(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)]d\tau\\
&-A'_{iFH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]\Big\} d\tau \geq 0,
\end{aligned}$$

which, in turn, holds under the sufficient condition

$$\int_0^T \alpha_H(\tau) \left\{ A_{iFH}(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)]d\tau \right.$$
$$\left. -A'_{iFH}(\tau) \left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]d\tau' \right] \right\} d\tau \geq 0. \tag{B.120}$$

Equation (B.120) holds under the sufficient condition that the function

$$G(\tau) \equiv \frac{A_{iFH}(\tau)}{\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]d\tau'}$$

is non-increasing because the term in curly brackets in (B.120) is the negative of the numerator of $G'(\tau)$. The function $G'(\tau)$ is non-increasing under the sufficient condition that the function

$$G_1(\tau) \equiv \frac{A'_{iFH}(\tau)}{A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)}$$

is non-increasing. Equation (B.39) for $(j, j') = (H, F)$ implies

$$\begin{aligned} G_1(\tau) &= \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\tau) + (a\sigma_{iF}^2 \bar{\lambda}_{iFF} - \kappa_{iF})A_{iFH}(\tau)}{A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)} \\ &= \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} + (a\sigma_{iF}^2 \bar{\lambda}_{iFF} - \kappa_{iF})\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}}{A_{iHe}\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)} + A_{iFe}}. \end{aligned}$$

Since $\bar{\lambda}_{iFH} \geq 0$, $\bar{\lambda}_{iFF} \leq 0$ and $\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing, $G_1(\tau)$ is non-increasing. ∎

C Model Estimation

C.1 Numerical Solution Method

We derive a system of 25 nonlinear scalar equations in the elements of the 5×5 matrix M. We adopt the exponential specification (5.6) and (5.7) for the functions $\{(\alpha_j(\tau), \theta_j(\tau))\}_{j=H,F}$, and set $T = \infty$. Using the exponential specification and $T = \infty$, we can compute the integrals involving $\{A_j(\tau)\}_{j=H,F}$ in the definition (B.5) of M by treating them as Laplace transforms of $\{A_j(\tau)\}_{j=H,F}$. These Laplace transforms can be computed by solving systems of linear equations with coefficients that are linear in the elements of M. The computation of the Laplace transforms does not require solving the ODE system (B.3) for $\{A_j(\tau)\}_{j=H,F}$, which would entail computing eigenvalues and eigenvectors of M.

We define the Laplace transform

$$\mathcal{A}_j(s) \equiv \int_0^\infty A_j(\tau) e^{-s\tau} d\tau$$

of $A_j(\tau)$, and

$$\mathcal{X}_j(s) \equiv \int_0^\infty X_j(\tau) e^{-s\tau} d\tau$$

of $X_j(\tau) \equiv A_j(\tau) A_j(\tau)^\top$. Multiplying (B.3) by $e^{-s\tau}$, taking integrals of both sides from zero to infinity, and using the property that the Laplace transform of $A_j'(\tau)$ is s times that of $A_j(\tau)$ (this property follows from integration by parts), we find

$$(sI + M)\mathcal{A}_j(s) = \frac{1}{s}\mathcal{E}_{ij} \to \mathcal{A}_j(s) = \frac{1}{s}(sI + M)^{-1}\mathcal{E}_{ij}, \tag{C.1}$$

where I denotes the 5×5 identity matrix. Multiplying (B.3) from the right by $A_j(\tau)^\top$, and adding to the resulting equation its transpose, we find

$$A_j'(\tau)A_j(\tau)^\top + A_j(\tau)A_j'(\tau)^\top + MA_j(\tau)A_j(\tau)^\top + A_j(\tau)A_j(\tau)^\top M^\top - \mathcal{E}_{ij}A_j(\tau)^\top - A_j(\tau)\mathcal{E}_j^\top = 0. \tag{C.2}$$

Multiplying (C.2) by $e^{-s\tau}$, taking integrals of both sides from zero to infinity, and using the definition of $X_j(\tau)$ and the property that the Laplace transform of $X_j'(\tau)$ is s times that of $X_j(\tau)$, we find

$$\left(\frac{s}{2}I + M\right)\mathcal{X}_j(s) + \mathcal{X}_j(s)\left(\frac{s}{2}I + M\right)^\top = \mathcal{E}_{ij}\mathcal{A}_j(s)^\top + \mathcal{A}_j(\tau)\mathcal{E}_{ij}^\top. \tag{C.3}$$

The solutions to (C.1) and (C.3) can be computed by solving systems of linear equations with coefficients that are linear in the elements of M. There are twenty-five scalar equations in the system for (C.1). There are fifteen scalar equations in the system for (C.3) because $\mathcal{X}_j(s)$ is a symmetric matrix. Equation (C.1) has a unique solution $\mathcal{A}_j(s)$ if $sI + M$ is invertible. Equation (C.3) is a Lyapunov equation and has a unique solution $\mathcal{X}_j(s)$ under the sufficient condition that the eigenvalues of $\frac{s}{2}I + M$ have positive real parts.

Using the Laplace transforms $(\mathcal{A}_j(s), \mathcal{X}_j(s))$ and the exponential specifications (5.6) and (5.7), we can compute the integrals involving $A_j(\tau)$ in the definition of M as

$$\int_0^\infty \theta_j(\tau)\mathcal{E}_{\beta j}A_j(\tau)^\top d\tau = -\theta_{j0}\mathcal{E}_{\beta j}\mathcal{A}_j'(\theta_{j1})^\top, \tag{C.4}$$

$$\int_0^\infty \alpha_j(\tau)A_j(\tau)A_j(\tau)^\top d\tau = \alpha_{j0}\mathcal{X}_j(\alpha_{j1}). \tag{C.5}$$

Deriving (C.4) requires using the property that the Laplace transform of $\tau A_j(\tau)$ is minus the derivative of the Laplace transform of $A_j(\tau)$. The derivative $\mathcal{A}'_j(s)$ can be computed as function of $\mathcal{A}_j(s)$ by differentiating (C.1):

$$\mathcal{A}_j(s) + (sI+M)\mathcal{A}'_j(s) = -\frac{1}{s^2}\mathcal{E}_{ij} \Rightarrow \mathcal{A}'_j(s) = -(sI+M)^{-1}\left(\mathcal{A}_j(s) + \frac{1}{s^2}\mathcal{E}_{ij}\right). \tag{C.6}$$

Using (C.4) and (C.5), together with

$$A_e = M^{-1}\left(\mathcal{E}_{iH} - \mathcal{E}_{iF}\right), \tag{C.7}$$

which follows from (B.1), we can write (B.5) as

$$\begin{aligned} M \equiv \Gamma^\top - a \Bigg[&\left(\theta_e \mathcal{E}_\gamma - \alpha_e M^{-1}\left(\mathcal{E}_{iH} - \mathcal{E}_{iF}\right)\right)\left(\mathcal{E}_{iH} - \mathcal{E}_{iF}\right)^\top \left(M^{-1}\right)^\top \\ &- \sum_{j=H,F}\left(\theta_{j0}\mathcal{E}_{\beta j}\mathcal{A}'_j(\theta_{j1})^\top + \alpha_{j0}\mathcal{X}_j(\alpha_{j1})\right)\Bigg]\Sigma\Sigma^\top. \end{aligned} \tag{C.8}$$

The right-hand side of (C.8) is a function of M, derived from (C.1), (C.3) and (C.6). Therefore, (C.8) forms a system of 25 nonlinear scalar equations in the 25 elements of M. Given M, we derive $A_j(\tau)$ by solving the ODE system (B.3), and we obtain A_e from (C.7). Given $A_j(\tau)$ and A_e, we solve for $C_j(\tau)$ and C_e from (B.2), (B.4) and (B.6).

We solve the system of 25 nonlinear scalar equations using a continuation algorithm.

- Step 0 of the algorithm solves the system for zero risk aversion $a^{(0)} = 0$. The solution is $M = \Gamma^\top$.
- Step $i+1$ of the algorithm solves the system for risk aversion $a^{(i+1)} = a^{(i)} + s^{(i+1)}$, where $a^{(i)}$ is risk aversion for step i and $s^{(i+1)}$ is a small step size. The solution $M^{(i)}$ in step i is used as initial condition for solving the system in step $i+1$. This ensures that the solution in step $i+1$ is found quickly and is close to the solution in step i.
- The algorithm ends when $a^{(i+1)} = a$.

If there are multiple solutions for M, the continuation algorithm picks the solution that converges to the unique solution $M = \Gamma^\top$ when risk aversion goes to zero.

C.2 MLE for Alternative Parametrizations

Table C.1 reports parameter estimates and standard errors when innovations to the currency demand factor are allowed to correlate with innovations to the bond demand factors. The correlations are small (correlation $\frac{\sigma_{\gamma,\beta H}}{\sqrt{\sigma^2_{\gamma,\beta H}+\sigma^2_{\gamma,\beta F}+\sigma^2_\gamma}} = -0.020$ between innovations to the currency demand factor and the home bond demand factor, and correlation $\frac{\sigma_{\gamma,\beta F}}{\sqrt{\sigma^2_{\gamma,\beta H}+\sigma^2_{\gamma,\beta F}+\sigma^2_\gamma}} = -0.185$ between innovations to the currency demand factor and the foreign bond demand factor). The estimates for the remaining parameters are similar to those in Table 1. The counterparts of the figures in Sections 5.3 and 5.4 (not reported here) are similar as well.

Parameter	Value	Std. Err.
σ_{iH}	1.144	0.077
σ_{iF}	0.867	0.056
$\sigma_{iH,iF}$	0.309	0.075
κ_{iH}	0.102	0.061
κ_{iF}	0.127	0.049
κ_{β}	0.059	0.058
κ_{γ}	0.041	0.116
$a\theta_e\kappa_{\gamma,iH}$	-142.1	109.5
$a\theta_e\kappa_{\gamma,iF}$	148.1	116.7
$a\theta_0\sigma_{\beta}$	887.9	173.6
$a\theta_e\sigma_{\gamma}$	760.9	395.2
$a\alpha_0$	5.977	3.143
$a\alpha_e$	71.76	34.59
$a\theta_e\sigma_{\gamma,\beta H}$	-15.23	109.8
$a\theta_e\sigma_{\gamma,\beta F}$	-143.4	123.2

Table C.1: Estimated model parameters for correlated currency and bond demand

Table C.2 reports parameter estimates and standard errors for $(\alpha_1, \theta_1) = (0.25, 0.5)$ and Table C.3 does the same for $(\alpha_1, \theta_1) = (0.3, 0.75)$. The estimates of all parameters except for $(a\theta_0\sigma_\beta, a\alpha_0)$ are similar to those in Table 1. The estimates for $(a\theta_0\sigma_\beta, a\alpha_0)$ become larger to compensate for the faster convergence of the functions $\{(\alpha_j(\tau), \theta_j(\tau))\}_{j=H,F}$ to zero. The counterparts of the figures in Sections 5.3 and 5.4 (not reported here) are similar.

Parameter	Value	Std. Err.
σ_{iH}	1.163	0.076
σ_{iF}	0.882	0.058
$\sigma_{iH,iF}$	0.326	0.083
κ_{iH}	0.143	0.059
κ_{iF}	0.136	0.046
κ_{β}	0.047	0.058
κ_{γ}	0.159	0.102
$a\theta_e\kappa_{\gamma,iH}$	-159.5	123.4
$a\theta_e\kappa_{\gamma,iF}$	193.6	135.2
$a\theta_0\sigma_{\beta}$	1079.0	222.3
$a\theta_e\sigma_{\gamma}$	983.2	477.4
$a\alpha_0$	14.68	6.659
$a\alpha_e$	80.47	38.74

Table C.2: Estimated model parameters for $(\alpha_1, \theta_1) = (0.25, 0.5)$

Parameter	Value	Std. Err.
σ_{iH}	1.165	0.076
σ_{iF}	0.888	0.058
$\sigma_{iH,iF}$	0.322	0.082
κ_{iH}	0.149	0.061
κ_{iF}	0.138	0.046
κ_{β}	0.055	0.058
κ_{γ}	0.158	0.104
$a\theta_e\kappa_{\gamma,iH}$	-162.9	124.5
$a\theta_e\kappa_{\gamma,iF}$	196.9	135.7
$a\theta_0\sigma_{\beta}$	1472.0	307.7
$a\theta_e\sigma_{\gamma}$	989.4	476.9
$a\alpha_0$	20.36	8.964
$a\alpha_e$	81.21	38.73

Table C.3: Estimated model parameters for $(\alpha_1, \theta_1) = (0.3, 0.75)$

Table C.4 reports parameter estimates and standard errors for the case where $K = 21$ and p_t includes US and German yields with maturities from one to ten years and the Dollar-Euro log exchange rate. The parameter estimates are similar to those in Table 1. The counterparts of the figures in Sections 5.3 and 5.4 (not reported here) are similar as well.

Parameter	Value	Std. Err.
σ_{iH}	1.180	0.075
σ_{iF}	0.940	0.065
$\sigma_{iH,iF}$	0.348	0.084
κ_{iH}	0.094	0.054
κ_{iF}	0.115	0.045
κ_{β}	0.033	0.050
κ_{γ}	0.129	0.088
$a\theta_e\kappa_{\gamma,iH}$	-145.4	120.6
$a\theta_e\kappa_{\gamma,iF}$	178.7	134.9
$a\theta_0\sigma_{\beta}$	729.1	155.7
$a\theta_e\sigma_{\gamma}$	946.4	472.2
$a\alpha_0$	4.569	2.737
$a\alpha_e$	80.12	39.55

Table C.4: Estimated model parameters using all yields from one to ten years

C.3 Sensitivity Analysis

Table C.5 shows the derivatives of the elements of the covariance matrix $A\hat{\Sigma}_{\Delta t}A^{\top}$ of innovations to the vector p_t in the MLE estimation with respect to the thirteen model parameters $(\sigma_{iH}, \sigma_{iF}, \sigma_{iH,iF}, \kappa_{iH}, \kappa_{iF}, \kappa_{\beta}, \kappa_{\gamma}, \theta_e\kappa_{\gamma,iH}, \theta_e\kappa_{\gamma,iF}, a\theta_0\sigma_{\beta}, a\theta_e\sigma_{\gamma}, a\alpha_0, a\alpha_e)$. The rows correspond to matrix elements and the columns to model parameters. Since $A\hat{\Sigma}_{\Delta t}A^{\top}$ is a

5×5 symmetric matrix, Table C.5 is 15×13. We denote elements of $A\hat{\Sigma}_{\Delta t}A^\top$ based on the corresponding observables. Consider, for example, the element in the third row and second column of $A\hat{\Sigma}_{\Delta t}A^\top$ (which is the same as the element in the second row and third column). That element is the covariance between innovations to the home ten-year yield and the foreign one-year yield, and is denoted by $y_H^{(10)}y_F^{(1)}$. We express the derivatives as elasticities by dividing by the matrix element and multiplying by the model parameter. We highlight in red the elasticities that are within 80% of the maximum elasticity in absolute value, and in yellow the elasticities that are within 50-80%. Table C.5 indicates which elements of $A\hat{\Sigma}_{\Delta t}A^\top$ are the main determinants of which parameters.

- The variance $y_H^{(1)}y_H^{(1)}$ of innovations to the home one-year yield determines σ_{iH} because it is sensitive to only σ_{iH}. Indeed, in the row corresponding to $y_H^{(1)}y_H^{(1)}$, there is a red in the column corresponding to σ_{iH} and no other reds or yellows.

- The variance $y_F^{(1)}y_F^{(1)}$ of innovations to the foreign one-year yield determines σ_{iF} because it is sensitive to only σ_{iF}. Indeed, in the row corresponding to $y_F^{(1)}y_F^{(1)}$, there is a yellow in the column corresponding to σ_{iF} and no other reds or yellows.

- The covariance $y_H^{(1)}y_F^{(1)}$ between innovations to the home and the foreign one-year yield determines $\sigma_{iH,iF}$ because it is sensitive to only $\sigma_{iH,iF}$. Indeed, in the row corresponding to $y_H^{(1)}y_F^{(1)}$, there is a red in the column corresponding to $\sigma_{iH,iF}$ and no other reds or yellows.

- The covariance $y_H^{(1)}y_H^{(10)}$ between innovations to the home one- and ten-year yield determines κ_{iH}. Indeed, in the row corresponding to $y_H^{(1)}y_H^{(10)}$, there is a yellow in the column corresponding to κ_{iH} and no other reds or yellows, except for a yellow in the column corresponding to σ_{iH}, which is determined above.

- The covariance $y_F^{(1)}y_F^{(10)}$ between innovations to the foreign one- and ten-year yield determines κ_{iF}. Indeed, in the row corresponding to $y_F^{(1)}y_F^{(10)}$, there is a yellow in the column corresponding to κ_{iH} and no other reds or yellows, except for a yellow in the column corresponding to σ_{iF}, which is determined above.

- The covariance $y_H^{(1)}e$ between innovations to the home one-year yield and the exchange rate determines $a\theta_e\kappa_{\gamma,iH}$. Indeed, in the row corresponding to $y_H^{(1)}e$, there is a red in the column corresponding to $a\theta_e\kappa_{\gamma,iH}$ and no other reds or yellows except in columns corresponding to parameters already determined and except for a yellow in the column corresponding to $a\theta_e\sigma_\gamma$.

- The covariance $y_F^{(1)}e$ between innovations to the foreign one-year yield and the exchange rate determines $a\theta_e\kappa_{\gamma,iF}$. Indeed, in the row corresponding to $y_F^{(1)}e$, there is a red in the column corresponding to $a\theta_e\kappa_{\gamma,iF}$ and no other reds or yellows except

Element	σ_{iH}	σ_{iF}	$\sigma_{iH,iF}$	κ_{iH}	κ_{iF}	κ_{β}	κ_{γ}	$a\theta_e\kappa_{\gamma,iH}$	$a\theta_e\kappa_{\gamma,iF}$	$a\theta_0\sigma_{\beta}$	$a\theta_e\sigma_{\gamma}$	$a\alpha_0$	$a\alpha_e$
$y_H^{(1)}\ y_H^{(1)}$	1.911	0.003	0.020	-0.156	-0.002	-0.000	-0.006	-0.041	0.017	0.035	0.074	-0.031	-0.034
$y_H^{(1)}\ y_H^{(10)}$	1.513	0.021	0.067	-0.418	-0.016	-0.016	-0.042	-0.152	0.076	0.282	0.309	-0.264	-0.154
$y_H^{(1)}\ y_F^{(1)}$	1.013	0.122	0.910	-0.082	-0.082	-0.000	0.014	0.025	0.035	0.059	-0.171	-0.053	0.072
$y_H^{(1)}\ y_H^{(10)}$	0.906	0.357	0.657	-0.068	-0.404	-0.023	0.080	0.033	0.133	0.513	-0.587	-0.445	0.276
$y_H^{(1)}\ e$	2.347	-0.096	-0.470	-0.260	0.072	-0.002	0.178	0.907	-0.404	0.059	-1.371	-0.046	0.089
$y_H^{(10)}\ y_H^{(10)}$	1.792	0.077	0.182	-0.738	-0.055	-0.097	-0.087	-0.303	0.171	1.937	0.580	-0.815	-0.285
$y_H^{(10)}\ y_F^{(1)}$	0.918	0.347	0.655	-0.375	-0.101	-0.024	0.070	0.101	0.073	0.507	-0.607	-0.443	0.279
$y_H^{(10)}\ y_H^{(10)}$	1.179	0.661	0.582	-0.406	-0.411	-0.074	0.100	0.066	0.124	2.090	-0.722	-0.991	0.328
$y_H^{(10)}\ e$	2.022	-0.313	-0.384	-0.498	0.100	-0.015	0.560	0.737	-0.487	0.436	-1.246	-0.261	0.079
$y_F^{(1)}\ y_F^{(1)}$	0.019	1.660	0.266	-0.002	-0.153	-0.000	-0.005	0.015	-0.037	0.030	0.059	-0.026	-0.023
$y_F^{(1)}\ y_H^{(10)}$	0.068	1.348	0.255	-0.009	-0.442	-0.013	-0.026	0.064	-0.141	0.239	0.238	-0.224	-0.102
$y_F^{(1)}\ e$	-0.675	2.639	-0.166	0.087	-0.271	-0.002	0.164	-0.453	1.054	0.032	-1.350	-0.023	-0.021
$y_H^{(10)}\ y_H^{(10)}$	0.256	1.531	0.410	-0.055	-0.771	-0.103	-0.044	0.154	-0.267	1.852	0.368	-0.747	-0.151
$y_H^{(10)}\ e$	-0.814	2.290	-0.092	0.113	-0.523	-0.006	0.521	-0.541	0.889	0.191	-0.964	-0.105	-0.165
$e\ e$	0.115	0.136	-0.039	-0.015	-0.012	-0.001	-0.364	0.076	0.068	0.021	2.082	-0.014	-2.077

Table C.5: Elasticities of the elements of $A\hat{\Sigma}_{\Delta t}A^{\top}$ with respect to the model parameters in columns corresponding to parameters already determined and except for a yellow in the column corresponding to $a\theta_e\sigma_\gamma$.

- The covariance $y_H^{(10)}e$ between innovations to the home ten-year yield and the exchange rate, and its foreign counterpart $y_F^{(10)}e$, determine κ_γ. Indeed, in the rows corresponding to $y_H^{(10)}e$ and $y_F^{(10)}e$, there is a red in the column corresponding to κ_γ and no other reds or yellows except in columns corresponding to parameters already determined and except for a yellow in the column corresponding to $a\theta_e\sigma_\gamma$.

- The variances $y_H^{(10)}y_H^{(10)}$ and $y_F^{(10)}y_F^{(10)}$ of innovations to the home and foreign ten-year yields, and the covariance $y_H^{(10)}y_F^{(10)}$ of innovations between these yields, determine κ_β, $a\theta_0\sigma_\beta$ and $a\alpha_0$. Indeed, in the rows corresponding to $y_H^{(10)}y_H^{(10)}$, $y_F^{(10)}y_F^{(10)}$ and $y_H^{(10)}y_F^{(10)}$, there are reds or yellows in the columns corresponding to κ_β, $a\theta_0\sigma_\beta$ and $a\alpha_0$ and no other reds or yellows except in columns corresponding to parameters already determined.

- The variance ee of innovations to the exchange rate, and the covariances between innovations to yields and the exchange rate, determine $a\theta_e\sigma_\gamma$ and $a\alpha_e$. Indeed, in the row corresponding to ee, there is a red in the columns corresponding to $a\theta_e\sigma_\gamma$ and ee and no other reds or yellows except in columns corresponding to parameters already determined. Moreover, in the column corresponding to $a\theta_e\sigma_\gamma$ there are several other yellows, while in the column corresponding to α_e there are no other reds or yellows.

The intuition for the determination of $a\alpha_e$ and $a\theta_e\sigma_\gamma$ is as follows. The effect of $a\alpha_e$ on the volatility of the exchange rate is particularly pronounced because higher demand elasticity of currency traders causes both larger under-reaction of the exchange rate to short rates and larger attenuation of currency demand shocks. Both effects lower the volatility of the exchange rate. By contrast, the effect of $a\theta_e\sigma_\gamma$ on the volatility of the exchange rate is less pronounced and more comparable to the effect on the covariance between the exchange rate and bond yields. This is because higher volatility of currency demand shocks causes larger under-reaction of the exchange rate to short rates, which tempers the effect of the demand shocks' higher volatility on the volatility of the exchange rate.

C.4 GMM

We use four sets of moments for GMM. A first set of moments concern one-year yields. They are the standard deviation of one-year yields $y_{jt}^{(1)}$ and of their annual change $\Delta y_{jt}^{(1)} \equiv y_{j,t+1}^{(1)} - y_{jt}^{(1)}$, and the standard deviation of the one-year yield differential $y_{Ht}^{(1)} - y_{Ft}^{(1)}$ between home and foreign.

A second set of moments concern the exchange rate. They are the standard deviation of the annual (log) exchange rate change $\Delta \log e_t \equiv \log e_{t+1} - \log e_t$; the correlation between $\Delta \log e_t$ and the one-year yield differential $y_{Ht}^{(1)} - y_{Ft}^{(1)}$; the correlation between $\Delta \log e_t$ and the annual change $\Delta y_{jt}^{(1)}$ in the home and foreign one-year yield; and the correlation between the five-year change in the exchange rate $\Delta^{(5)} \log e_t \equiv \log e_{t+5} - \log e_t$ and the five-year yield differential $y_{Ht}^{(5)} - y_{Ft}^{(5)}$.

A third set of moments concern yields across all maturities up to twenty years. They are the standard deviation of yields $y_{jt}^{(\tau)}$ and of their annual change $\Delta y_{jt}^{(\tau)} \equiv y_{j,t+1}^{(\tau)} - y_{jt}^{(\tau)}$; the correlation between the annual changes $\Delta y_{jt}^{(1)}$ in one-year yields and $\Delta y_{jt}^{(\tau)}$ in all other yields; and the standard deviation of yield differentials $y_{Ht}^{(\tau)} - y_{Ft}^{(\tau)}$ for all maturities.

A final set of moments concern trading volume. They are the trading volume of US government bonds with maturities between zero and three years, and with maturities between eleven and thirty years, as a fraction of total US government bond trading volume (denoted by $\tilde{V}_H(0 \leq \tau \leq 3)$ and $\tilde{V}_H(11 \leq \tau \leq 30)$, respectively).

The total number of target moments is $N = 12 + 7(\mathcal{N}_T - 1)$, where $\mathcal{N}_T$ is the number of bond maturities (we subtract one to not double-count the one-year maturity). With maturities going from one to twenty years in annual increments, $\mathcal{N}_T$ equals twenty and the number of target moments is 145 (=12 + 7 × 19). We refer to the twelve moments that do not depend on maturity as scalar.

We estimate the model by choosing the model parameters that minimize

$$\mathcal{L} \equiv \sum_{n=1}^{N} w_n (\hat{m}_n - m_n)^2, \tag{C.9}$$

the weighted sum of squared differences between the empirical moments $\{\hat{m}_n\}_{n=1,..,N}$ and their model-implied counterparts $\{m_n\}_{n=1,..,N}$. We set the weights w_n to one for scalar moments and to $\frac{1}{\mathcal{N}_T}$ for moments that depend on maturity, so that each type of moment receives the same weight (for moments corresponding to the one-year maturity, we use $1 + \frac{1}{\mathcal{N}_T}$).

To compute the model-implied moments of exchange rates and bond yields, we first compute the unconditional covariance and autocovariance of the state vector q_t. Integrating the dynamics (2.7) of q_t between $-\infty$ and t, we find

$$q_t - \bar{q} = \int_{-\infty}^{t} e^{-\Gamma(t-s)} \Sigma dB_s. \tag{C.10}$$

Equation (C.10) implies that the unconditional covariance matrix of q_t is

$$\mathbb{C}\mathrm{ov}(q_t, q_t^\top) = \int_{-\infty}^{t} e^{-\Gamma(t-s)} \Sigma \Sigma^\top e^{-\Gamma^\top (t-s)} ds \equiv \hat{\Sigma}. \tag{C.11}$$

Differentiating (C.11) with respect to t and noting that the derivative is zero, we find

$$\Gamma\hat{\Sigma} + \hat{\Sigma}\Gamma^\top = \Sigma\Sigma^\top, \tag{C.12}$$

which is a Lyapunov equation and has a unique solution $\hat{\Sigma}$ because the eigenvalues of Γ have positive real parts. The unconditional autocovariance matrix of q_t is

$$\begin{aligned}\mathbb{C}\mathrm{ov}(q_t, q_{t'}^\top) &= \int_{-\infty}^{t} e^{-\Gamma(t-s)}\Sigma\Sigma^\top e^{-\Gamma^\top(t'-s)} ds \\ &= \left[\int_{-\infty}^{t} e^{-\Gamma(t-s)}\Sigma\Sigma^\top e^{-\Gamma^\top(t-s)} ds\right] e^{-\Gamma^\top(t'-t)} \\ &= \hat{\Sigma} e^{-\Gamma^\top(t'-t)},\end{aligned} \tag{C.13}$$

for $t' > t$, where the last step in (C.13) follows from (C.11).

Bond yields and log exchange rates in the model are affine functions of the state vector q_t. The covariance between two such affine functions $Xq_t + X_0$ and $Yq_{t'} + Y_0$ for 1×5 constant vectors (X, Y), scalars (X_0, Y_0), and $t' > t$ is

$$\mathbb{C}\mathrm{ov}(Xq_t + X_0, Yq_{t'} + Y_0) = X\mathbb{C}\mathrm{ov}(q_t, q_{t'}^\top)Y^\top. \tag{C.14}$$

Table C.4 reports parameter estimates and standard errors for GMM. The GMM point estimates are similar to the MLE ones in Table 1. GMM also delivers estimates for (a_1, θ_1), which are similar to the values $(0.15, 0.3)$ that we use in Section 5. The counterparts of the figures in Sections 5.3 and 5.4 (not reported here) are similar except for the confidence intervals for the UIP regressions and the exchange-rate responses to monetary policy. These are significantly wider under GMM because of the higher standard errors of the estimates for $(\theta_e\kappa_{\gamma,iH}, \theta_e\kappa_{\gamma,iF}, a\theta_e\sigma_\gamma, a\alpha_e)$.

C.5 Variance Decomposition

To compute the variance decomposition, we must account for the correlation between the instantaneous innovations to the home short rate i_{Ht} and the foreign short rate i_{Ft}, which arises because of the off-diagonal element $\sigma_{iH,iF}$ of the diffusion matrix Σ. Attributing all the return variance that arises because of the Brownian motion B_{iHt} to i_{Ht} amounts to attributing to i_{Ht} all the variance that arises because of its correlated part with i_{Ft}. To attribute the return variance due to the correlated part symmetrically across i_{Ht} and i_{Ft}, we rotate the Brownian motions (B_{iHt}, B_{iFt}) to new versions (B_{RiHt}, B_{RiFt}) that render the diffusion matrix symmetric. We then compute the fraction of return variance that arises because of each Brownian motion in the rotated vector $B_{Rt} \equiv (B_{RiHt}, B_{RiFt}, B_{\gamma t}, B_{\beta Ht}, B_{\beta Ft})^\top$ and attribute it to the corresponding element of $q_t \equiv (i_{Ht}, i_{Ft}, \gamma_t, \beta_{Ht}, \beta_{Ft})^\top$.

Parameter	Value	Std. Err.
σ_{iH}	1.429	0.148
σ_{iF}	0.751	0.140
$\sigma_{iH,iF}$	1.054	0.083
κ_{iH}	0.126	0.030
κ_{iF}	0.090	0.020
κ_{β}	0.050	0.009
κ_{γ}	0.134	0.102
$a\theta_e\kappa_{\gamma,iH}$	-267.1	550.4
$a\theta_e\kappa_{\gamma,iF}$	252.1	527.7
$a\theta_0\sigma_{\beta}$	648.9	80.27
$a\theta_e\sigma_{\gamma}$	762.7	1067.0
$a\alpha_0$	4.740	3.302
$a\alpha_e$	73.38	106.3
α_1	0.144	0.031
θ_1	0.374	0.014

Table C.6: Estimated Model Parameters using GMM

We denote the rotated diffusion matrix by Σ_R. Since Σ_R is symmetric and the product of Σ times a rotation matrix R,

$$\Sigma_R\Sigma_R = \Sigma_R\Sigma_R^\top = (\Sigma R)(\Sigma R)^\top = \Sigma R R^\top \Sigma^\top = \Sigma\Sigma^\top.$$

Therefore, $\Sigma_R = (\Sigma\Sigma^\top)^{\frac{1}{2}}$.

Since the log exchange rate and bond yields are affine in q_t, the same calculation as in (5.1) implies that the surprise component of a currency or bond log return over a horizon Δt is $A_r \int_t^{t+\Delta t} e^{-\Gamma(t+\Delta t-s)}\Sigma_R dB_{Rs}$, where the 1×5 vector A_r is endogenously determined from the model parameters. The variance of the return is $A_r\hat{\Sigma}_{\Delta t}A_r^\top$, where

$$\hat{\Sigma}_{\Delta t} \equiv \int_t^{t+\Delta t} e^{-\Gamma(t+\Delta t-s)}\Sigma_R\Sigma_R^\top e^{-\Gamma^\top(t+\Delta t-s)}ds = \int_t^{t+\Delta t} e^{-\Gamma(t+\Delta t-s)}\Sigma\Sigma^\top e^{-\Gamma^\top(t+\Delta t-s)}ds$$

is the covariance matrix (computed in Section 5.1) of the innovations to q_t over horizon Δt. The return variance that arises because of the j'th element of the vector B_{Rt} is $A_r\hat{\Sigma}_{\Delta t,j}A_r^\top$, where

$$\hat{\Sigma}_{\Delta t,j} \equiv \int_t^{t+\Delta t} e^{-\Gamma(t+\Delta t-s)}\Sigma_R\mathfrak{J}_j\Sigma_R^\top e^{-\Gamma^\top(t+\Delta t-s)}ds$$

and $\mathfrak{J}_j$ is a matrix whose terms are zero except for the term (j,j), which is one. The matrix $\hat{\Sigma}_{\Delta t,j}$ satisfies the Lyapunov equation

$$\Gamma\hat{\Sigma}_{\Delta t,j} + \hat{\Sigma}_{\Delta t,j}\Gamma^\top = \Sigma_R\mathfrak{J}_j\Sigma_R^\top - e^{-\Gamma\Delta t}\Sigma_R\mathfrak{J}_j\Sigma_R^\top e^{-\Gamma^\top\Delta t}.$$

The fraction of return variance generated by the j'th element of the vector B_{Rt} is

$$\frac{A_r \hat{\Sigma}_{\Delta t,j} A_r^\top}{A_r \hat{\Sigma}_{\Delta t} A_r^\top}.$$

C.6 Correlations

The top left panel of Figure C.1 shows the correlation between changes to the one-year and τ-year home bond yields as function of τ. The top right panel shows the same correlation for foreign bond yields. The bottom left panel shows the correlation between changes to the log exchange rate and the τ-year home and foreign bond yields as function of τ. The bottom right panel shows the correlation between changes to the τ-year home and foreign bond yields as function of τ. All changes are quarterly. The empirical correlations are the red circles or brown diamonds. The model-implied correlations are the blue solid or purple dashed lines. The black dashed or black dotted lines show the model-implied correlations when arbitrageurs are risk-neutral. The model-implied correlations can be computed as described in (C.14).

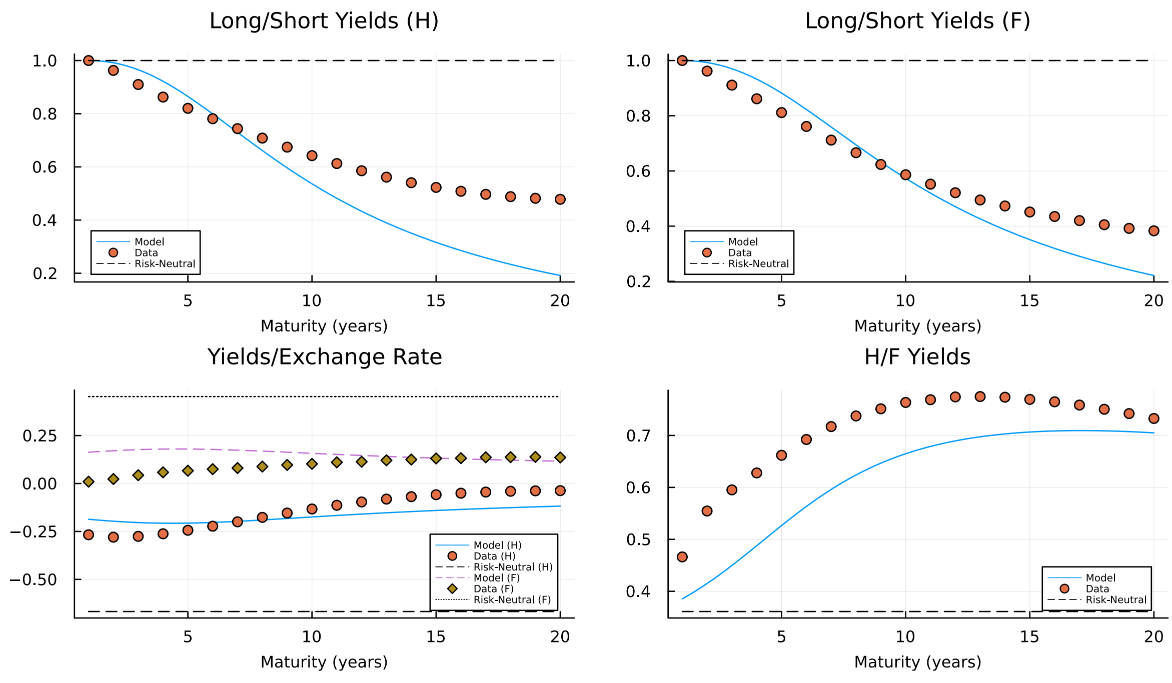

Figure C.1: Correlations between the exchange rate and bond yields

The empirical correlations are close to the model-implied ones. The correlations between long-maturity (10- to 20-year) home and foreign bond yields are around 70%. They are higher than the correlations when arbitrageurs are risk-neutral, which are around 35%. When arbitrageurs are risk-neutral, a country's bond yields move only because of that country's short rate. Therefore, the correlation between bond yields across countries is equal to that between short rates. When arbitrageurs are risk-averse, the correlation between bond yields is influenced also by the bond demand factors, which generate positive comovement for the reasons explained in Section 5.3.

The correlations between changes to the exchange rate and long-maturity (10- to 20-year) bond yields are positive for foreign yields, negative for home yields, and smaller than 20% in absolute value. They are significantly smaller in absolute value than the correlations between home and foreign bond yields for the reasons explained in Section 5.3. They are smaller in absolute value than the correlations when arbitrageurs are risk-neutral because the bond demand factors become a significant driver of long-maturity bond yields but do not generate comovement between yields and the exchange rate.

The correlations between short- and long-maturity bond yields are equal to one when arbitrageurs are risk-neutral because a country's bond yields move only because of that country's short rate. When arbitrageurs are risk-averse, the correlations become significantly smaller than one because the bond demand factors become a significant driver of long-maturity bond yields while generating only small comovement between short- and long-maturity yields.

C.7 Predictive Regressions

Bilson (1981) and Fama (1984) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{e_t}{e_{t+\Delta\tau}}\right) = a_{\text{UIP}} + b_{\text{UIP}}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the rate of foreign currency depreciation over horizon $\Delta\tau$. The independent variable is the foreign-minus-home $\Delta\tau$-year yield differential. Bilson (1981) and Fama (1984) assume that the horizon $\Delta\tau$ is short (monthly). Chinn and Meredith (2004) perform the same regression for longer horizons. The coefficient b_{UIP} of this regression depends on second moments of bond yields and log exchange rates, and can be computed as described in (C.14).

Lustig, Stathopoulos, and Verdelhan (2019) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{F,t+\Delta\tau}^{(\tau-\Delta\tau)}e_{t+\Delta\tau}}{P_{Ft}^{(\tau)}e_t}\right) - \frac{1}{\Delta\tau}\log\left(\frac{P_{H,t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_{Ht}^{(\tau)}}\right) = a_{\text{LSV}} + b_{\text{LSV}}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the return over horizon $\Delta\tau$ of the hybrid CCT constructed using bonds with maturity τ. The independent variable is the foreign-minus-home $\Delta\tau$-year yield differential. Since log bond prices are affine functions of the state vector q_t, the coefficient b_{LSV} of this regression can be computed as described in (C.14).

Lloyd and Marin (2020) and Chernov and Creal (2021) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{e_t}{e_{t+\Delta\tau}}\right) = a_{\text{UIP}\ell s} + b_{\text{UIP}\ell}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + b_{\text{UIPs}}\left[\left(y_{Ft}^{(\tau_2)} - y_{Ft}^{(\tau_1)}\right) - \left(y_{Ht}^{(\tau_2)} - y_{Ht}^{(\tau_1)}\right)\right] + e_{t+\Delta\tau}.$$

The dependent variable is the rate of foreign currency depreciation over horizon $\Delta\tau$. The independent variables are the foreign-minus-home $\Delta\tau$-year yield differential and the foreign-minus-home slope differential between years τ_1 and τ_2. The coefficients b_{UIPs} and $b_{\text{UIP}\ell}$ of this regression can be computed as described in (C.14).

Fama and Bliss (1987) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{j,t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_{jt}^{(\tau)}}\right) - y_{jt}^{(\Delta\tau)} = a_{\text{FB}} + b_{\text{FB}}\left(f_{jt}^{(\tau-\Delta\tau,\tau)} - y_{jt}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the log return over horizon $\Delta\tau$ of the country-j bond with maturity τ in excess of the $\Delta\tau$-year spot rate (yield). The independent variable is the slope of the country-j term structure as measured by the difference between the forward rate between maturities $\tau-\Delta\tau$ and τ, and the $\Delta\tau$-year spot rate. Since log bond prices are affine functions of the state vector q_t, and the forward rate is

$$f_{jt}^{(\tau-\Delta\tau,\tau)} = -\frac{\log\left(\frac{P_{jt}^{(\tau)}}{P_{jt}^{(\tau-\Delta\tau)}}\right)}{\Delta\tau},$$

the coefficient b_{FB} of this regression can be computed as described in (C.14).

Campbell and Shiller (1991) perform the regression

$$y_{j,t+\Delta\tau}^{(\tau-\Delta\tau)} - y_{jt}^{(\tau)} = a_{\text{CS}} + b_{\text{CS}}\frac{\Delta\tau}{\tau-\Delta\tau}\left(y_{jt}^{(\tau)} - y_{jt}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the change over horizon $\Delta\tau$ in the yield of a country-j bond with initial maturity τ. The independent variable is the difference between the country-j spot rates for maturities τ and $\Delta\tau$, normalized so that b_{CS} is equal to one under the EH. The coefficient b_{CS} of this regression can be computed as described in (C.14).

C.8 Monetary Policy Transmission

The top left and top right panels of Figure C.2 show, respectively, how the cut to the home short rate described in Section 5.4 affects arbitrageur bond holdings at time zero as function of maturity, and how it affects their currency holdings over time. The bottom left and right panels show the same for the cut to the foreign short rate. Holdings of home bonds are shown in blue and of foreign bonds in red. Holdings are expressed as fraction of US GDP.

Following the cut to the home short rate, aggregate holdings of home bonds by arbitrageurs increase by 0.205% of GDP, as can be derived by integrating the blue line over maturities. The sharpest increase occurs for maturities around three years. Arbitrageur holdings of foreign bonds remain essentially unchanged—they decrease by 0.001%

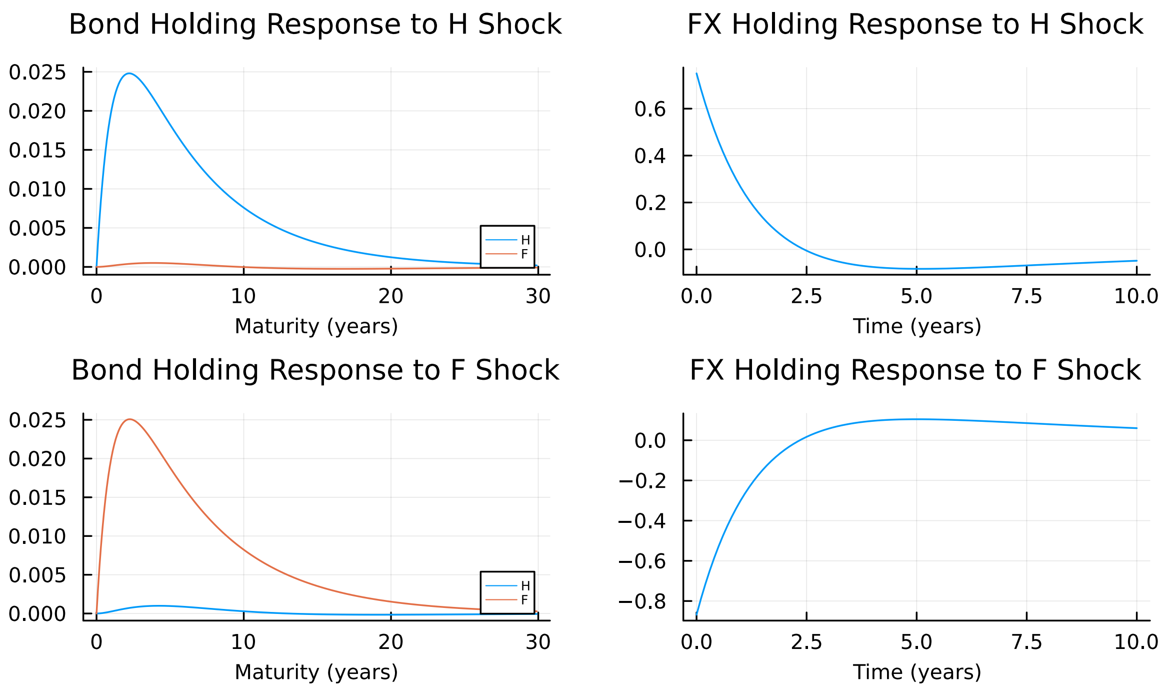

Figure C.2: Conventional monetary policy and arbitrageur holdings – Short rate cut

of GDP. Arbitrageur currency holdings increase by 0.751% of GDP following the cut, and then decline gradually to their pre-cut value. The responses to the foreign short rate's cut have similar magnitudes (with currency holdings decreasing following the cut).

The top left and top right panels of Figure C.3 show, respectively, how the QE purchases in the home country described in Section 5.4 affect arbitrageur bond holdings at time zero as function of maturity, and how they affect their currency holdings over time. The bottom left and right panels show the same for the QE purchases in the foreign country. The coloring and units are as in Figure C.2.

Following the purchases of home bonds, aggregate holdings of home bonds by arbitrageurs decrease by 7.160% of GDP. Arbitrageurs thus sell to the central bank bonds worth 71.6% (=7.16/10) of all the bonds that the central bank purchases (which are assumed to be worth 10% of GDP). The maturities that arbitrageurs sell the most are around five years. Arbitrageur holdings of foreign bonds increase by 1.105% of GDP, as they seek to partly replace the home bonds that they sell to the central bank by foreign bonds. Arbitrageur currency holdings increase by 0.844% of GDP following the cut, and then decline gradually to their pre-cut value. The responses to the purchases of foreign bonds have similar magnitudes (with currency holdings decreasing following the cut).

Figures C.4 and C.5 are the counterparts of Figures 4 and 5 for subsamples. For each of Figures C.4 and C.5, the left two columns correspond to the subsample 06/1986-12/2007 and the right two columns to the subsample 01/1999-04/2021. Our results are reasonably stable when viewed in conjunction with the confidence intervals in Figures 4

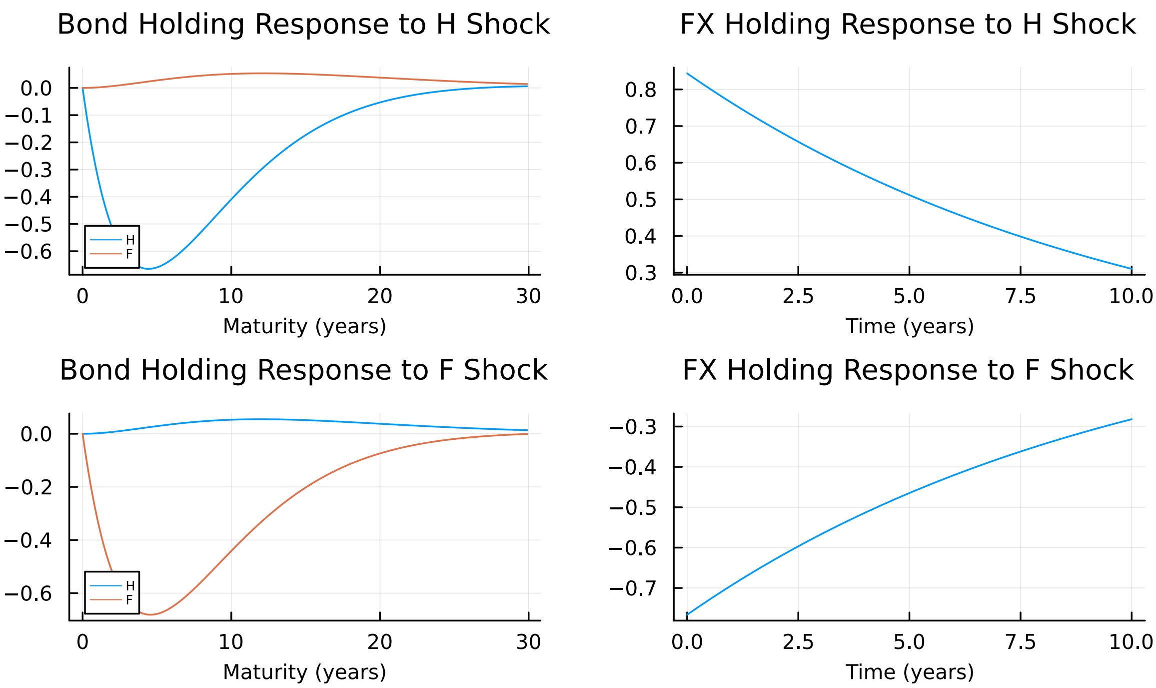

Figure C.3: Unconventional monetary policy and arbitrageur holdings – Bond purchases

and 5.

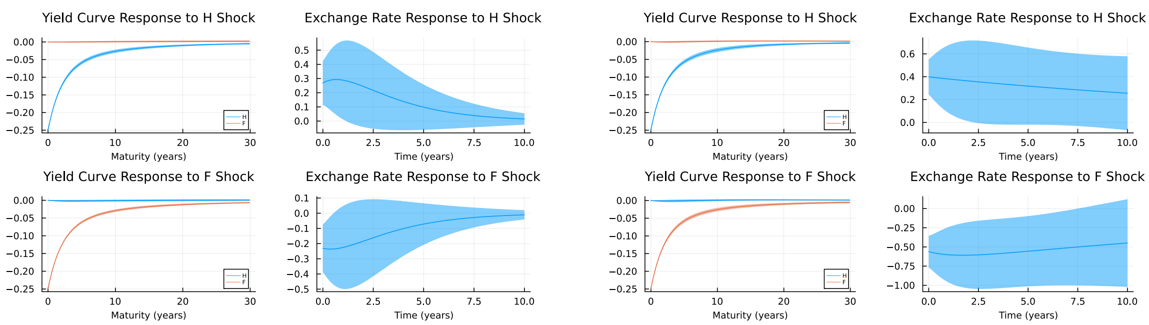

Figure C.4: Conventional monetary policy – Subsamples

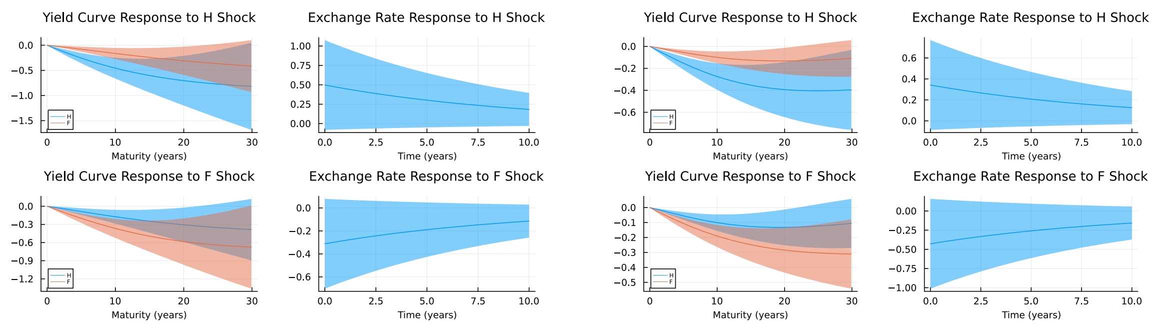

Figure C.5: Unconventional monetary policy – Subsamples

Figure C.6 shows additional comparative statics of the effects of QE. We vary the slope parameter α_0 of bond demand, the slope α_e of currency demand, and the correlation parameter $\sigma_{iH,iF}$ between the home and the foreign short rate. All parameters except for the one that varies are set to their estimated values, except in the case of $\sigma_{iH,iF}$ where we also vary σ_{iF} to keep the standard deviation $\sqrt{\sigma_{iH,iF}^2 + \sigma_{iF}^2}$ of innovations to the foreign short rate constant. The estimated value of the parameter that varies corresponds to one in the x-axis by normalizing the units. The figure shows the ten-year yield and the exchange rate. Results in the top row are for α_0, in the middle row for α_e, and in the bottom row for $\sigma_{iH,iF}$.

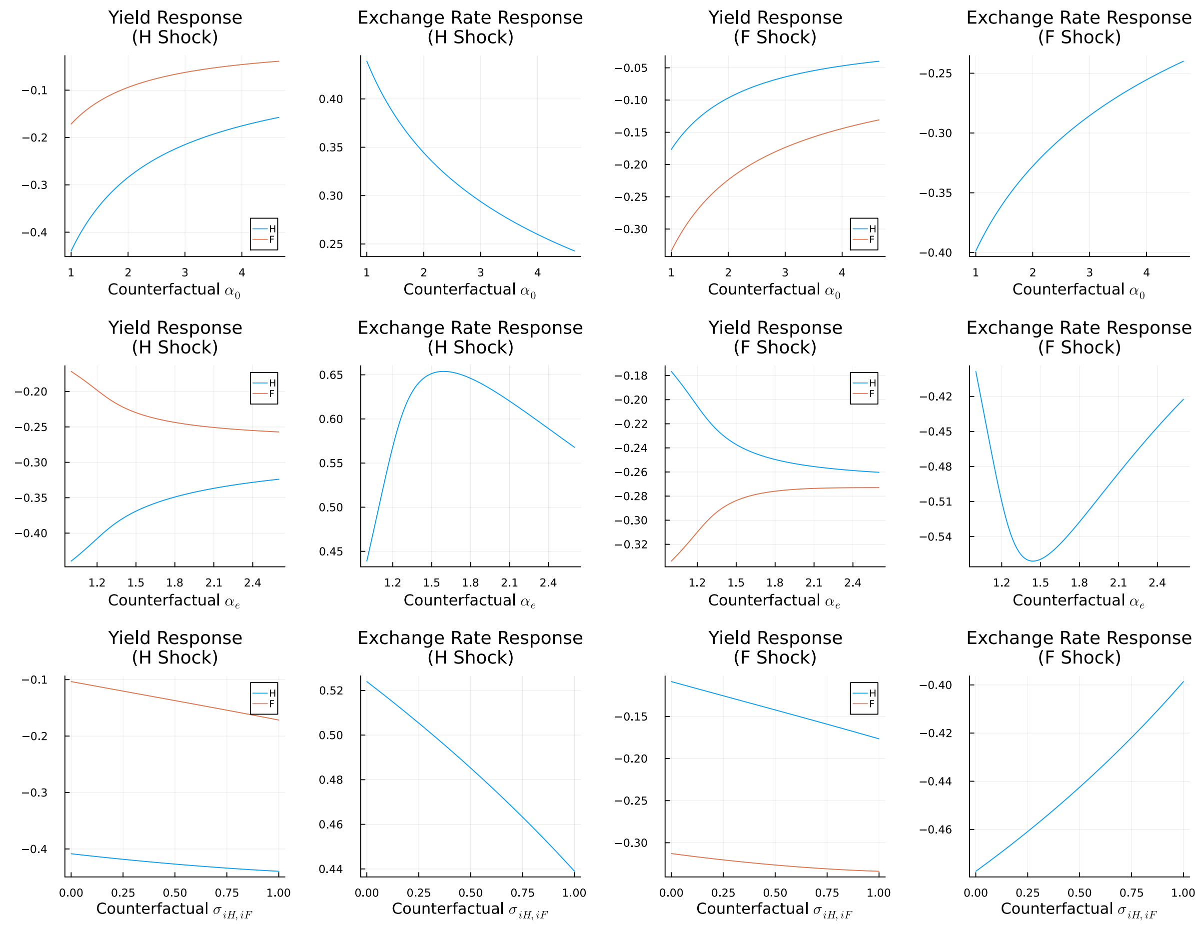

Figure C.6: Additional comparative statics of effects of QE

When α_0 increases, QE has weaker effects on domestic bond yields. This is because bond investors require a smaller price increase to sell domestic bonds to the central bank. As a result, arbitrageurs sell fewer domestic bonds to the central bank, and adjust less their hedges in currency and foreign bonds. Because of the arbitrageurs' reduced trading, QE has weaker effects on the exchange rate and foreign bond yields.

When α_e increases, QE has weaker effects on domestic bond yields. This is because arbitrageurs can adjust their hedge in currency with smaller price impact, and thus become more willing to sell domestic bonds to the central bank. Because arbitrageurs adjust more their currency hedge, they also adjust more their foreign-bond hedge. As a result, QE has stronger effects on foreign bond yields. The effects of QE on the exchange rate are non-monotone because on the one hand the adjustment to the currency hedge is larger but on the other hand price impact in the currency market is smaller.

When $\sigma_{iH,iF}$ increases, QE has stronger effects on domestic and foreign bond yields, and weaker effects on the exchange rate. The intuition for foreign bond yields is that arbitrageurs are better able to use foreign bonds to replace the domestic bonds that they sell to the central bank, causing foreign bond yields to decrease by more. The intuition for the exchange rate is that it becomes less responsive to short-rate shocks. (In the limit of perfectly correlated short rates, the exchange rate is fixed.) Therefore, currency becomes less valuable as a hedge, and the exchange rate responds less to QE. The intuition for domestic yields is as follows. On the one hand, domestic yields should decrease by less because arbitrageurs are better able to use foreign bonds to replace the domestic bonds that they sell to the central bank. On the other hand, domestic yields should decrease by more because currency becomes less valuable as a hedge. Either effect can dominate, and the second one does under our estimated parameter values.

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